Fakultät für Mathematik und Informatik
Friedrich-Schiller-Universität Jena
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Numerik Partieller Differentialgleichungen I (B.Sc.), and Theory and Numerical Analysis of Partial Differential Equations I (M.Sc.)

Exercise Set 4

Submission:

If you wish to submit one of the marked (highlighted with $*$) exercises from Exercise Sheet 4, it must be submitted in class on **03.06.2026** or sent via email before the class starts. Please write your full name and matriculation number at the top right of your submission.

Contact:

For any questions regarding the course or exercises, please send an email to Christos Pervolianakis (christos.pervolianakis@uni-jena.de).

Exercise 1(*): [Minimum angle condition] Prove that the shape regularity is equivalent to requiring that there exists a $\theta_0 > 0$ independent of $\mathcal{T}_h$ such that $\theta_K \geq \theta_0$, $\forall\, K \in \mathcal{T}_h$, where θ_K is the minimum interior angle of K.

Hint. *For both directions, use the trigonometric form of the inradius. For the $\Leftarrow$, use a contradiction argument.*

Exercise 2(*): Let $m = 2$, then define $\mathcal{P} = \mathcal{Q}_2$ with $\dim(\mathcal{Q}_2) = 9$, and assume that $\mathcal{N}_2 = \{N_1, \ldots, N_9\}$ where $N_i(v) = v(z_i), i = 1, 2, 3, 4$, where z_1, z_2, z_3, z_4 are the vertices of K and $N_i(v) = v(z_i)$, $i = 5, 6, 7, 8$ where z_5, z_6, z_7, z_8, are the midpoints of edges with endpoints (z_1, z_2), (z_2, z_3), (z_3, z_4), (z_4, z_1). Furthermore, $N_9(v) = v(z_9)$, with z_9 the centroid of K. Prove that $\mathcal{N}_2$ determines $\mathcal{Q}_2$.

Hint. *Use similar arguments to Example 3.3.7.*

Exercise 3: Let $K \subset \mathbb{R}^2$ be any triangle and $\mathcal{P}_m$, $m \geq 1$, denote the set of all polynomials in two variables of degree at most m, i.e.,

$$\mathcal{P}_m = \left\{ \sum_{|a| \leq m} c_a x^a \ : \ c_a \in \mathbb{R} \right\}.$$

Prove that $\dim(\mathcal{P}_m) = \frac{1}{2}(m+1)(m+2)$.

Exercise 4: Let $\mathcal{I}_K$ be the P_1 Lagrange interpolation operator on a triangle K. Prove that

$$\|\mathcal{I}_K v\|_{\mathcal{C}(K)} \leq \|v\|_{\mathcal{C}(K)} \quad \forall v \in \mathcal{C}(K).$$

Exercise 5: Let $\widehat{K}$, K affinely equivalent triangles, see Remark 3.4.4. Recall the Gagliardo–Nirenberg–Ladyzhenskaya inequality

$$\|\widehat{u}\|_{L^4(\widehat{K})} \leq C_{GNL}(\widehat{K}) \|\widehat{u}\|^{1/2}_{L^2(\widehat{K})} \|\widehat{\nabla}\widehat{u}\|^{1/2}_{L^2(\widehat{K})} \quad \text{for} \quad \widehat{u} \in H^1(\widehat{K}),$$

where the constant C_{GNL} may depend on $\widehat{K}$. Use a scaling argument to prove that there exist a constant C_1, independent of K, such that

$$\|u\|_{L^4(K)} \leq C_1 \|u\|_{H^1(K)}, \ \ \forall\, u \in H^1(K).$$

Exercise 6: Let $\widehat{K}$ the triangle with vertices $(0,0)$, $(0,1)$, $(1,0)$. Let $\widehat{\chi} \in \mathbb{P}_1(\widehat{K})$, then due to the norm equivalence of the norms in the finite dimensional spaces, there exists a constant $C_{\widehat{K}}$ depending on $\widehat{K}$ such that

$$\|\widehat{\chi}\|_{L^\infty(\widehat{K})} \leq C_{\widehat{K}} \|\widehat{\chi}\|_{L^2(\widehat{K})}, \quad \forall\, \widehat{\chi} \in \mathbb{P}_1(\widehat{K}).$$

Let K affinely equivalent to $\widehat{K}$ triangle, see Remark 3.4.3. Use a scaling argument to prove that there exist a constant C_1, independent of K, such that

$$\|\chi\|_{L^\infty(K)} \leq C_1\, h_K^{-1} \|\chi\|_{L^2(K)}, \;\; \chi \in \mathbb{P}_1(K).$$