FAKULTÄT FÜR MATHEMATIK UND INFORMATIK
FRIEDRICH-SCHILLER-UNIVERSITÄT JENA
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Numerik Partieller Differentialgleichungen I (B.Sc.), and Theory and Numerical Analysis of Partial Differential Equations I (M.Sc.)

Exercise Set 5

Submission:
If you wish to submit one of the marked (highlighted with $*$) exercises from Exercise Sheet 5, it must be submitted in class on **17.06.2026** or sent via email before the class starts. Please write your full name and matriculation number at the top right of your submission.

Contact:
For any questions regarding the course or exercises, please send an email to Christos Pervolianakis (christos.pervolianakis@uni-jena.de).

Exercise 1(*): Let K the triangle with vertices $(0,0)$, $(1,0)$, $(-1,\mu)$ with $0 < \mu \ll 1$. Consider the function $u(x,y) = x^2$. Compute the $\mathbb{P}_1$ Lagrange interpolant as we have defined in lecture notes, see Definition 3.3.4. Then, show that

$$\|\nabla(v - \mathcal{I}_K v)\|_{L^2(K)} \geq \frac{1}{\mu}|v|_{H^2(K)},$$

where $|v|^2_{H^2(K)} = \sum_{\alpha=2} \|D^\alpha v\|^2_{L^2(K)}$, i.e., the H^2-seminorm on K.

Hint. *Compute the interpolant directly. See also the Example 3.3.9.*

Exercise 2(*): Let $\widehat{\Omega}$, Ω affinely equivalent bounded domains in $\mathbb{R}^d$, see Remark 3.4.2. Let $\mathcal{P}$ be a finite dimensional subspace of $L^p(\Omega) \cap L^q(\Omega)$ where $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Prove that, there exists a constant C depending only on $\widehat{\mathcal{P}}$, $\widehat{\Omega}$, p, q, γ, such that for all $v \in \mathcal{P}$, we have Prove that

$$\|v\|_{L^p(\Omega)} \leq Ch^{d\left(\frac{1}{p} - \frac{1}{q}\right)}\|v\|_{L^q(\Omega)}.$$

Hint. *This is a special case of Lemma 3.4.5. Use the same arguments of that lemma and derive the result.*

Exercise 3: Prove Theorems 3.5.1 and 3.5.2

Exercise 4: Let Ω be a Lipschitz domain in $\mathbb{R}^d$. Prove that exists constants C_1, C_2, that depends on Ω, such that

$$\|v\|_{L^p(\partial\Omega)} \leq C_1\|v\|_{L^p(\Omega)} + C_2\|\nabla v\|^{1/p}_{L^p(\Omega)}\|v\|^{1-1/p}_{L^p(\Omega)},$$

for all $1 \leq p < \infty$ and all $v \in W^{1,p}(\Omega)$.

Exercise 5:
1. Let $k \geq 1$, $1 \leq p \leq \infty$. We define the reference simplex on $\mathbb{R}^d$, as

$$\widehat{K} := \left\{(\widehat{x}_1, \ldots, \widehat{x}_d) \in (0,1)^d \ : \ \sum_{i=1}^d \widehat{x}_i \leq 1\right\}.$$

Why the

$$\widehat{c}_{k,p} := \sup_{\widehat{v}\in\mathbb{P}_{k,d}} \frac{\|\widehat{\nabla}\widehat{v}\|_{L^p(\widehat{K})}}{\|\widehat{v}\|_{L^p(\widehat{K})}}$$

is finite? Here $\mathbb{P}_{k,p}$ is composed of $d-$variate polynomial functions $p : \mathbb{R}^d \to \mathbb{R}$ of total degree at most k, i.e.,

$$\mathbb{P}_{k,p} := \mathsf{span}\{x_1^{\alpha_1}\cdots x_d^{\alpha_d},\ 0 \leq \alpha_1,\ldots,\alpha_d \leq k,\ \alpha_1+\cdots+\alpha_d \leq k\}.$$

2. Let K be a simplex in $\mathbb{R}^d$ and let ρ_K denote the diameter of its largest inscribed ball. Show that

$$\|\nabla v\|_{L^p(K)} \leq \widehat{c}_{k,p}\frac{\sqrt{2}}{\rho_K}\|v\|_{L^p(K)}, \quad \forall v \in \mathbb{P}_{k,d}(K).$$

Exercise 6: Let W be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_W$. Let U be a subset of W and V a closed subspace of W. Let

$$\beta := \inf_{u\in U}\sum_{v\in V}\frac{|\langle u,v\rangle_W|}{\|u\|_W\|v\|_V}.$$

1. Prove that $0 \leq \beta \leq 1$.
2. Prove that

$$\beta = \inf_{u\in U}\frac{\|\Pi_V(u)\|_W}{\|u\|_W}m$$

where Π_V is the orthogonal projection onto V.

3. Prove the estimate

$$\|u-\Pi_V(u)\|_W \leq (1-\beta^2)^{1/2}\|u\|_W.$$