Fakultät für Mathematik und Informatik
Friedrich-Schiller-Universität Jena
Dr. Christos Pervolianakis

Numerik Partieller Differentialgleichungen I (B.Sc.), and Theory and Numerical Analysis of Partial Differential Equations I (M.Sc.)

Exercise Set 6

Submission:

If you wish to submit one of the marked (highlighted with $*$) exercises from Exercise Sheet 6, it must be submitted in class on **08.07.2026** or sent via email before the class starts. Please write your full name and matriculation number at the top right of your submission.

Contact:

For any questions regarding the course or exercises, please send an email to Christos Pervolianakis (christos.pervolianakis@uni-jena.de).

Exercise 1(*): Let $\Omega = [0,1]$ a compact interval on $\mathbb{R}$. We define the following weak formulation. We seek $u \in U$ such that

$$\int_\Omega u' v \, dx = \int_\Omega f v \, dx, \quad \forall\, v \in V,$$

where $'$ denotes the derivative. The spaces $V := L^2(\Omega)$ and

$$U := \{u \in H^1(\Omega) \,:\, u(0) = 0\}.$$

We also assume that $f \in L^2(\Omega)$, given. Prove that the above problem is well–posed (existence and uniqueness).

Hint. *Use the Banach–Necas–Babuska theorem.*

Exercise 2(*): Let $\alpha \in (0,1)$. Consider the Lagrange $\mathbb{P}_1$ shape functions $\varphi_1(x) = 1 - x$ and $\varphi_2(x) = x$ on $\Omega = [0,1]$. Let a sequence of continuous functions $\{u_n\}_{n\in\mathbb{N}\setminus\{0\}}$ on Ω defined by $u_n(x) = n^\alpha - 1$ for $0 \le x \le 1/n$ and $u_n(x) = x^{-\alpha} - 1$, otherwise.

1. Prove that the sequence ins uniformly bounded in $L^p(\Omega)$ for all p such that $p\alpha < 1$.
2. Compute it Lagrange interpolant $\mathcal{I}_\Omega(u_n)$. Is the intepolant stable in L^p norm? Equivalently, exists a constant c_1 such that $\|\mathcal{I}_\Omega(u_n)\|_{L^p(\Omega)} \le c_1 \|u_n\|_{L^p(\Omega)}$?
3. Is the interpolant $\mathcal{I}_K$ stable in any L^r norm with $r \in [1,\infty)$?

Exercise 3: Let $\Omega \subset \mathbb{R}^d$ a Lipschitz domain and $\mathcal{F} : W^{k+1,p}(\Omega) \to \mathbb{R}$ a bounded linear functional, i.e., $\mathcal{F} \in (W^{k+1,p}(\Omega))^*$. Further, we assume that $\mathcal{F}(p) = 0, \forall\, \mathbb{P}_{k,d}$, where the real vector space $\mathbb{P}_{k,d}$ is composed of $d-$variate polynomial functions $p : \mathbb{R}^d \to \mathbb{R}$ of total degree at most k. Then,

$$\mathbb{P}_{k,p} := \mathsf{span}\{x_1^{\alpha_1} \cdots x_d^{\alpha_d},\ 0 \le \alpha_1, \ldots, \alpha_d \le k,\ \alpha_1 + \cdots + \alpha_d \le k\}.$$

Prove that there exists a constant C such that

$$|\mathcal{F}(v)| \le C \mathrm{diam}(\Omega)^{k+1} \|\mathcal{F}\|_{(W^{k+1,p}(\Omega))^*} |v|_{W^{k+1,p}(\Omega)}.$$

Exercise 4: Let K a simplex in $\mathbb{R}^d$ and $\mathcal{P} := \mathbb{P}_{1,d}$, as defined in Exercise 3. Recall that the face of K opposite to z_i is denoted by F_i. We define the Crouzeix–Raviart element is defined as

$$\sigma_i^{\mathrm{CR}}(p) := \frac{1}{|F_i|} \int_{F_i} p \, dS, \quad \forall\, i = 0, \dots, d.$$

Set $\mathcal{N} = \{\sigma_i^{\mathrm{CR}}\}_{i=1,\dots,d}$. Prove that $(K, \mathcal{P}, \mathcal{N})$ is a finite element.

Exercise 5: Let $(K, \mathcal{P}, \mathcal{N})$ a finite element with $\dim \mathcal{P} = m$. We define the so–called dof-based norm (dof=degrees of freedom, i.e., the $N_1, \dots, N_m$), as $|||v||| := \max_{1 \le i \le m} |N_i(v)|, \ \ \forall\, v \in \mathcal{P}$. Prove that there exists a constant C such that

$$C\|v\|_{L^p(K)} \le |K|^{1/p} |B_K^{-1}| |||v||| \le C^{-1} \|v\|_{L^p(K)},$$

where $|B|$ denotes the matrix norm induced by the Euclidean vector norm in $\mathbb{R}^d$, i.e.,

$$|B| = \sup_{0 \ne x \in \mathbb{R}^d} \frac{|Bx|}{|x|}, \ \ |x| = \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2}.$$