Theorie und Numerik partieller Differentialgleichungen I

L_2 Projection in 1D

Let $I = [0, L],\ L > 0$, be an interval and let $N+1$ point $\{x_i\}_{i=0}^N$ to define the partition

$$0 = x_0 < x_1 < \ldots < x_{N_1} < x_N = L, \tag{1}$$

that subdivides the interval I into N subintervals $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, N$ of length $h_i = x_i - x_{i-1}$. Define, the space V_h, of continuous piecewise linear functions by

$$V_h = \{v\ :\ v \in \mathcal{C}(I),\ :\ v|_{I_i} \in \mathbb{P}_1(I_i)\}, \tag{2}$$

where $\mathbb{P}_1(I_i)$ denotes the space of linear functions on I_1. Let $\{\phi_i\}_{i=0}^N$, a basis functions of V_h such that $\phi_i(x_j) = \delta_{ij}$, where $\delta_{ij} = 1$ if and only if $i = j$, and otherwise is zero. Since $\{\phi_i\}_{i=0}^N$ form a basis of V_h, every function $v_h \in V_h$, may be written as linear combination of the basis functions, i.e.,

$$v_h = \sum_{i=0}^N \alpha_i \phi_i(x), \quad \alpha = (\alpha_0, \ldots, \alpha_N)^T \in \mathbb{R}^{N+1}.$$

The functions ϕ_i are given by

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i}, & \text{if } x \in I_i, \\ \frac{x_{i+1} - x}{h_{i+1}}, & \text{if } x \in I_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

1 L_2 projection

Given a function $f \in L_2(I)$, we are we seek function $\pi_h f \in V_h$ such that

$$\int_I (f - \pi_h f)\chi \, dx = 0, \quad \forall \chi \in V_h.$$

The latter defines the a projection of function f onto V_h, since the difference $f - \pi_h f$ is required to be orthogonal to all functions of V_h. But how good is the L_2 in the approximating f? The following result gives the answer.

The L_2 projection $\pi_h f$ of f is the best approximation on V_h with respect to L_2-norm, i.e.,

$$\|f - \pi_h f\|_{L_2(I)} \le \|f - \chi\|_{L_2(I)}, \quad \forall \chi \in V_h.$$

Moreover, if $f \in H^2(I)$, the error can be estimated as follows,

$$\|f - \pi_h f\|_{L_2(I)} + h\|(f - \pi_h f)'\|_{L_2(I)} \le C\, h^2 \|f''\|_{L_2(I)}, \tag{3}$$

where $\|v\|_{L_2(I)} = \left(\int_I v^2 \, dx\right)^{1/2}$. The constant C is independent of h and $h = \max_{1 \le i \le N} h_i$.

1.1 Derivation of a Linear System of Equations

Since $\pi_h f \in V_h$, may be written as linear combination of the basis functions, i.e.,

$$\pi_h f = \sum_{i=0}^N \xi_i \phi_i(x), \quad \xi = (\xi_0, \ldots, \xi_N)^T \in \mathbb{R}^{N+1}.$$

Given a function $f \in L_2(I)$, in order to determine its L_2-projection $\pi_h f$ on V_h, we need to solve a linear system of equations. In fact, using its definition with $\chi = \phi_i$, $i = 1, \ldots, N$, we have

$$\int_I f\phi_i \, dx = \int_I \left(\sum_{j=0}^N \xi_j \phi_j \right) \phi_i \, dx = \sum_{j=1}^N \int_I \phi_j \phi_i \, dx, \quad i = 0, \ldots, N.$$

If further, we define the matrix $\mathcal{M} = \{m_{ij}\}_{i,j=0}^{N}$, with elements

$$m_{ij} := \int_I \phi_j \phi_i \, dx,$$

we get the following linear system for the determination of the coefficients $\xi_j,\ j = 0, \ldots, N$,

$$\mathcal{M}\,\xi = b,$$

where $b_i = \int_I f\phi_i \, dx,\ i = 0, \ldots, N$. Solving the latter linear system, we get the coefficient vector ξ.

To compute the vector b we need to employ a quadrature rule. For this exercise we may use the Simpson formula that integrates exactly polynomial up to order 2.

1.2 Simpson's rule

Given interval $[a, b]$, with midpoint $c = (a + b)/2$, the Simpson's rule for the computation of the $\int_I g \, dx,\ g \in \mathcal{C}([a, b])$, is

$$\int_I g \, dx = (b - a)\frac{g(a) + 4g(c) + g(b)}{6}.$$

1.3 Programming exercises

Exercise 1 *Let $I = [0, 1]$ and $f(x) = x^2,\ x \in I$. Write a program that*

1. *computes the L_2-projection $\pi_h f \in V_h$ of f,*
2. *determines the experimental order of accuracy in L_2 and H^1 norm, (see hint), and*
3. *plot your results and compare with the function.*

Hint *Compute the norms of (3) for two different natural numbers $N_1 < N_2$, then the experimental order of convergence with N_1, N_2, is given by*

$$p(N_1, N_2) = \frac{\ln\left(\frac{\mathcal{E}(N_2)}{\mathcal{E}(N_1)}\right)}{\ln\left(\frac{N_1}{N_2}\right)}, \tag{4}$$

where first in (4) take $\mathcal{E}(N) := \|f - \pi_h f\|_{L_2(I)},\ h = 1/N$. See that $p(N_1, N_2) \approx 2$. Next, take $\mathcal{E}(N) := \|(f - \pi_h f)'\|_{L_2(I)},\ h = 1/N$. See that $p(N_1, N_2) \approx 1$.

Exercise 2 *Take $f(x) = \arctan((x - 1/2)/\zeta),\ x \in I$. Run the program with different values of $\zeta = 0.1, 0.01$.*