Numerical approximation of a system of ODEs by explicit Euler

Christos Pervolianakis
Institute of Mathematics and Computer Science,
University of Jena, 07743, Jena, Germany
christos.pervolianakis@uni-jena.de

Problem statement

We seek function $y : [a, b] \to \mathbb{R}^d$, with $d \geq 1$, solution of the initial value problem

$$y'(t) = f(t, y(t)), \qquad t \in [a, b], \tag{1}$$
$$y(a) = y_0, \tag{2}$$

where the function $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$, and the real numbers a, b with $a < b$ and o $y_0 \in \mathbb{R}^d$, are given.

Explicit Euler

Given a natural number M, we define the uniform mesh of $[a, b]$ with fixed mesh step $h = \frac{b-a}{M}$ and nodes the points $t^n = a + nh$, for $n = 0, \dots, M$. The explicit Euler compute the approximation $Y^n \in \mathbb{R}^d$ of $y(t^n)$ given by

$$Y^n = Y^{n-1} + h\, f(t^{n-1}, Y^{n-1}), \quad n = 0, \dots, M, \tag{3}$$
$$Y^0 = y_0. \tag{4}$$

Notice that every element of the sequence $\{Y^n\}_{n=0}^N$ is a vector of dimension d.

Exercise 1

Write a program that computes the approximations of the explicit Euler $\{Y^n\}_{n=0}^M$ for the following initial value problem,

$$y'(t) = A\, y(t), \qquad t \in [0, 1], \tag{5}$$
$$y(0) = (1, 0)^T, \tag{6}$$

where the matrix A defined as

$$A := \begin{bmatrix} -1 & -e^{-2t} \\ e^{2t} & 1 \end{bmatrix}. \tag{7}$$

The exact solution of (5)-(6) is $y(t) = \left(e^{-t}\cos(t)\,, e^{t}\sin(t)\right)^T,\ t \in [0, 1]$. To check if the code is correct, one way is to compute the approximation error, i.e., given a natural number M, the approximation error is given by

$$\mathcal{E}(M) := \max_{0 \leq n \leq M} \max_{1 \leq i \leq d} |Y_i^n - y_i(t^n)|\,. \tag{8}$$

Now, to compute the approximation error (8), take two different natural numbers $M_1 < M_2$, and compute the experimental error for M_1, M_2, which is given by

$$p(M_1, M_2) = \frac{\ln\left(\frac{\mathcal{E}(M_2)}{\mathcal{E}(M_1)}\right)}{\ln\left(\frac{M_1}{M_2}\right)}. \tag{9}$$

Conclude that $p(M_1, M_2) \approx 1$.

Hint

The error for $M = 20$ and $M = 40$, are

$$\mathcal{E}(20) = 0.110972$$
$$\mathcal{E}(40) = 5.632574e - 02.$$