Numerical approximation of an ODEs by explicit multistep methods

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Problem statement

We seek function $y : [a, b] \to \mathbb{R}^d$, with $d \geq 1$, solution of the initial value problem

$$y'(t) = f(t, y(t)), \qquad t \in [a, b], \tag{1}$$
$$y(a) = y_0, \tag{2}$$

where the function $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$, and the real numbers a, b with $a < b$ and o $y_0 \in \mathbb{R}^d$, are given.

Explicit Euler

Given a natural number M, we define the uniform mesh of $[a, b]$ with fixed mesh step $h = \frac{b-a}{M}$ and nodes the points $t^n = a + nh$, for $n = 0, \ldots, M$.

For the explicit multistep methods we have the following parameters,

- the number of the initial approximations $k \in \mathbb{N}$,
- the coefficients $\{\alpha_j\}_{j=0}^{k-1} \in \mathbb{R}$,
- the coefficients $\{\beta_j\}_{j=0}^{k-1} \in \mathbb{R}$,

where in general $\alpha_k = 1$ with $|\alpha_0| + |\beta_0| > 0$ such that the method be k-step. An explicit k-step method for the numerical approximation of the initial value problem (1)-(2) is described for the coefficient (constant) $\{\alpha_j\}_{j=0}^{k-1}, \{\beta_j\}_{j=0}^{k-1}$ and produces vectors $\{Y^{n+k}\}_{n=0}^{M-k}$ which are given for $n = 0, \ldots, M - k$ by

$$\begin{aligned} &Y^0, Y^1, \ldots, Y^{k-1} \qquad , \\ &Y^{n+k} = -\sum_{j=0}^{k-1} \alpha_j Y^{n+j} + h \sum_{j=0}^{k-1} \beta_j f(t^{n+j}, Y^{n+j}), \quad n = 0, \ldots, N - k. \end{aligned} \tag{3}$$

If p is the order of k-step method, then the following are hold due to *Dahlquist (1959)*,

- $p \leq k + 1$ if k odd,
- $p \leq k + 2$ if k even.

We know that the k-step method of maximum order are implicit. For a stable explicit k-step method, we always have that $p \leq k$. We will recall the explicit k-step methods for which hold that $p = k$. In fact, the methods with this property, are called Adams-Bashforh and we will write them as $AB(k)$. For the $AB(k)$ we have the coefficients α_j, β_j for $j = 0, \ldots, k - 1$,

$$\alpha_{k-1} = -1, \quad \alpha_j = 0, \quad j = 0, \ldots, k - 2, \tag{4}$$

where the coefficients α_j, $j = 0, \ldots, k - 1$ are independent of k. On the other hand, the coefficients β_j, $j = 0, \ldots, k - 1$ depend on k and for $k = 2, 3, 4$ they defined as

$$\begin{aligned} k = 2: &\quad \beta_0 = -\frac{1}{2}, \quad \beta_1 = \frac{3}{2} \\ k = 3: &\quad \beta_0 = \frac{5}{12}, \quad \beta_1 = -\frac{4}{3}, \; \beta_2 = \frac{23}{12} \\ k = 4: &\quad \beta_0 = -\frac{9}{24}, \; \beta_1 = \frac{37}{24}, \; \beta_2 = -\frac{59}{24}, \; \beta_3 = \frac{55}{24} \end{aligned} \tag{5}$$

Numerical approximation

Step 1: Set $Y^0 := y_0$.
Step 2: Calculate the remaining initial conditions $\{Y^n\}_{n=1}^{k-1}$ using a different numerical method (can be one-step or even multistep) of order $q \geq p-1$ where p is the order that k-step method have. If we use numerical methods of order $q < p-1$, then we will "pollute" the order of convergence and the order will be q (instead of p, as the previous choice).
Step 3: Last, for $n = 0, \ldots, M-k$, we compute the vectors $Y^{n+k} \in \mathbb{R}^d$ from

$$Y^{n+k} = -\sum_{j=0}^{k-1} \alpha_j \, Y^{n+j} + h \sum_{j=0}^{k-1} \beta_j \, f(t^{n+j}, Y^{n+j}).$$

Notice that every element of the sequence $\{Y^n\}_{n=0}^N$ is a vector of dimension d.

Exercise 1

Write an code that computes the vectors $\{Y^n\}_{n=0}^N$ of $k-$step method $AB(k)$ for $k = 2, 3, 4$, for the initial value problem,

$$y'(t) = \frac{1}{10} y(t), \qquad t \in [0, 1], \tag{6}$$

$$y(0) = 1. \tag{7}$$

The exact solution of (6)-(7) is give by $y(t) = e^{\frac{t}{10}}$, $t \in [0, 1]$. To check if you have solve the exercise correctly, compute the approximation error for $AB(k)$, $k = 2, 3, 4$. The approximation error, given a natural number M, is calculated by

$$\mathcal{E}(M) := \max_{0 \leq n \leq M} |Y^n - y(t^n)|. \tag{8}$$

Compute the error (8) for two different natural numbers $M_1 < M_2$, in order to compute the approximating order convergence for M_1, M_2, which is defined as

$$p(M_1, M_2) = \frac{\ln\left(\frac{\mathcal{E}(M_2)}{\mathcal{E}(M_1)}\right)}{\ln\left(\frac{M_1}{M_2}\right)}. \tag{9}$$

Conclude that $p(M_1, M_2) \approx k$.

Case 1

Implement the $AB(2)$, where for Y^1 use the explicit method of Euler.

Case 2

Implement the $AB(3)$, where Y^1 and Y^2 use the classical Runge-Kutta method of 4 stages and 4 order.

Step 3

Implement the $AB(4)$, where for Y^1 and Y^2 classical Runge-Kutta method of 4 stages and 4 order and for the Y^3 use $AB(3)$ of the previous case.

Hint

The errors for $M = 20$ and $M = 40$, are

Case 1

$$\mathcal{E}(20) = 1.48930418e-05$$
$$\mathcal{E}(40) = 3.73238008e-06.$$

Case 2

$$\mathcal{E}(20) = 4.63750416e-09$$
$$\mathcal{E}(40) = 6.13538997e-10.$$

Case 3

$$\mathcal{E}(20) = 2.78008061e-10$$
$$\mathcal{E}(40) = 1.75344184e-11.$$

Exercise 2

Write a code that computes the approximations $\{Y^n\}_{n=0}^N$ of $k-$step $AB(k)$ for $k = 2, 3, 4$, where we have defined above, for the initial value problem

$$y'(t) = f(t, y(t)), \qquad t \in [0, 1], \tag{10}$$

$$y(0) = (1, 0)^T, \tag{11}$$

with $y(t) = (y_1(t), y_2(t))^T$ and $f_1(t, y(t)) = -y_1(t) - e^{-2t} y_2(t)$ and $f_2(t, y(t)) = y_2(t) + e^{2t} y_1(t)$. The exact solution of (10)-(11) is given by $y(t) = \left(e^{-t}\cos(t), e^{t}\sin(t)\right)^T, \ t \in [0, 1]$. To check if you have solve the exercise correctly, compute the approximation error for $AB(k)$, $k = 2, 3, 4$. The approximation error, given a natural number M, is calculated by

$$\mathcal{E}(N) := \max_{0 \le n \le N} \max_{1 \le i \le d} |Y_i^n - y_i(t^n)|. \tag{12}$$

Compute the error (8) for two different natural numbers $M_1 < M_2$, in order to compute the approximating order convergence for M_1, M_2, which is defined as

$$p(M_1, M_2) = \frac{\ln\left(\frac{\mathcal{E}(M_2)}{\mathcal{E}(M_1)}\right)}{\ln\left(\frac{M_1}{M_2}\right)}. \tag{13}$$

Conclude that $p(M_1, M_2) \approx k$.

Hint

The errors for $M = 20$ and $M = 40$, are

Case 1

$$\mathcal{E}(20) = 0.00683023$$
$$\mathcal{E}(40) = 0.00169556.$$

Case 2

$$\mathcal{E}(20) = 4.26925802e - 04$$
$$\mathcal{E}(40) = 5.35211044e - 05.$$

Case 3

$$\mathcal{E}(20) = 3.25498660e - 05$$
$$\mathcal{E}(40) = 2.08844545e - 06.$$