The Monge–Ampère equation

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Some accompanying problems for the lecture in Winter 2022/23 at U Jena

Problem 1. Let $u \in C(\mathbb{R}^n)$ be convex and let $\varphi \in C^2(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at $x_0 \in \mathbb{R}^n$. Prove that $D^2\varphi(x_0) \geq 0$.
Hint: Let $p \in \mathbb{R}^n$ define a supporting hyperplane to u at x_0. Use the second-order Taylor expansion of φ to conclude that $p = \nabla\varphi(x_0)$. Then conclude with the expansion that $D^2\varphi(x_0) \geq 0$.

Problem 2. Prove that the definition of viscosity subsolutions from the lecture remains the same if we replace the assumption that $u - \varphi$ has a maximum at x_0 by the stronger condition that $u - \varphi$ has a *strict* maximum at x_0.
Hint: Given φ, add $r|x - x_0|^2$ to φ, use the assumed "stronger" solution property and let $r \to 0$ at the end.

Problem 3. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, let $f \in C(\Omega)$ with $f \geq 0$ be given, and let $u \in C(\overline{\Omega})$ be a viscosity subsolution to $\det D^2 u = f$ in Ω. Suppose $v \in C^2(\Omega) \cap C(\overline{\Omega})$ is convex and satisfies $\det D^2 v \leq g$ in Ω for some $g \in C(\Omega)$ with $g < f$ in Ω. Prove that

$$\max_{\overline{\Omega}}(u - v) = \max_{\partial\Omega}(u - v).$$

Formulate an analogous (symmetric) property for viscosity supersolutions.
Hint: Assume for contradiction that the max is assumed inside Ω.

Problem 4 (jets)**.** The second-order super-jet of a continuous function u at $x \in \mathbb{R}^n$ is defined as

$$J^{2,+}u(x) := \{(p, X) \in \mathbb{R}^n \times \mathbb{S}^n : u(x + z) \leq u(x) + p \cdot z + \frac{1}{2} z^T X z + o(|z|^2) \text{ as } z \to 0\}.$$

(1) Prove that

$$J^{2,+}u(x) = \{(\nabla\varphi(x), D^2\varphi(x)) : \varphi \in C^2(\mathbb{R}^n) \text{ and } u - \varphi \text{ has a local max at } x\}.$$

(2) Prove that $u \in C(\Omega)$ is a subsolution to the Monge–Ampère equation if and only if

$$\inf_{(p,A)\in J^{2,+}u(x)} \det D^2 A - f(x) \geq 0 \quad \text{for all } x \in \Omega.$$

Problem 5 (geometric-arithmetic mean inequality)**.** Prove that any real numbers $x_1, \dots, x_n \geq 0$ satisfy $\prod_j x_j^{1/n} \leq n^{-1} \sum_j x_j$. Prove that equality holds if and only if all x_j coincide. *Hint:* Concavity of log and Jensen's inequality.

Problem 6 (elementary)**.** Let $A, B \in \mathbb{S}$. (1) Prove that $A = \sum_j \lambda_j v_j \otimes v_j$ for the eigenvalues λ_j and unit eigenvectors v_j. (2) Prove that $A : B = \operatorname{tr}(AB)$.

Problem 7. Decide which of the following functions $u : \mathbb{B}_1(0) \to \mathbb{R}$ over the unit ball $B_1(0) \subseteq \mathbb{R}^n$ are viscosity sub/supersolution to $\det D^2 u = 0$:

- $u(x) = 1$
- $u(x) = |x|^2$ (squared Euclidean norm)
- $u(x) = |x_1|$ (absolute value of the first component)
- $u(x) = |x|$ (Euclidean norm)
- $u(x) = -\cos(x_1)$
- $u(x) = -\cos(50x_1)$

Problem 8. Let $\mathcal{A}$ be a family of affine functions over $\mathbb{R}^n$. Prove that $\sup \mathcal{A}$ is a convex.

Problem 9. Let $u : \mathbb{R}^n \to \mathbb{R}$ be convex. Prove that u is locally Lipschitz continuous. *Instruction (if needed):* To show Lipschitz continuity near $x_0 \in \Omega$, let $B_{2r}(x_0) \subseteq \Omega$ be an open ball with $x, y \in B_r(x_0)$ and define $z := x + \alpha(x - y)$ with $\alpha = r/(2|x - y|)$. Show $x = (1+\alpha)^{-1}z + \alpha(1+\alpha)^{-1}y$ and use this result to first estimate $f(x) - f(y) \leq (\alpha+1)^{-1}(f(z) - f(y))$ and then establish the Lipschitz bound. Prove the estimate for $|f(x) - f(y)|$ by interchanging the roles of x, y.

Problem 10. Let $u : \mathbb{R}^n \to \mathbb{R}$ have the property that for any $x \in \mathbb{R}^n$ and any $\varphi \in C^2(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at x, there holds $D^2\varphi(x) \geq 0$. Prove that u must be convex.
Hint: You may restrict your attention to $n = 1$. Assume for contradiction that u is not convex and construct a non-convex φ touching from above at x.