Computational PDEs II

(Theorie und Numerik partieller Differentialgleichungen II)

D. Gallistl

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Contents

1. Galerkin method

§1. Closed range theorem and Banach–Babuška–Nečas lemma

We want to characterize isomorphisms between certain Banach spaces. We start by recalling a version of the Hahn–Banach theorem from linear functional analysis.

Theorem 1.1 (Hahn–Banach)**.** *Let $M \subseteq X$ be a subspace of the normed linear space $(X, \|\cdot\|_X)$ and let $f \in M^*$. Then there exists $F \in X^*$ such that $F|_M = f$ and $\|F\|_{X^*} = \|f\|_{M^*}$.*

Proof. This is taught in any class on linear functional analysis. ■

As a consequence, we note the following fundamental separation property.

Theorem 1.2 (separation)**.** *Let $M \subseteq X$ be a closed subspace of the Banach space X and let $z \in X \setminus M$ be a point outside M. Then there exists $F \in X^*$ with $\|F\|_{X^*} = 1$ that satisfies $F|_M = 0$ and $F(z) = \operatorname{dist}(z, M)$.*

Proof. We construct the linear functional f on $\tilde{M} = M + \operatorname{span}\{z\}$ by

$$f(y + \alpha z) = \alpha \operatorname{dist}(z, M) \quad \text{for any } y \in M, \alpha \in \mathbb{R}.$$

We compute

$$|f(y + \alpha z)| \leq |\alpha| \operatorname{dist}(z, M) \leq |\alpha| \, \|z + \alpha^{-1} y\|_X = \|\alpha z + y\|_X$$

which shows continuity of f, that is, $f \in \tilde{M}^*$ and $\|f\|_{\tilde{M}^*} \leq 1$. By the definition of the distance and the closedness of M, we have that, given any $\varepsilon > 0$, there exists $y_\varepsilon \in M$ such that $\|z - y_\varepsilon\|_X \leq (1 + \varepsilon) \operatorname{dist}(z, M)$ such that $f(z - y_\varepsilon) \geq (1 + \varepsilon)^{-1} \|z - y_\varepsilon\|_X$. Thus, $\|f\|_{\tilde{M}^*} \geq 1$. We now apply the Hahn–Banach theorem to $\tilde{M}$ and f, which shows the existence of the claimed extension F. ■

We use the notation $\langle f, v \rangle = f(v)$.

Definition 1.3 (annihilator, polar set)**.** Let X be a Banach space with a subspace $V \subseteq X$ and let $U \subseteq X^*$ be a subspace of its dual. We define the *annihilator* of V by

$$V^\circ := \{f \in X^* : \langle f, v \rangle = 0 \text{ for all } v \in V\} \subseteq X^*$$

and the *polar set* of U by

$${}^\circ U := \{x \in X : \langle u, x \rangle = 0 \text{ for all } u \in U\} \subseteq X.$$

♦

We have the following elementary property.

Lemma 1.4 (characterization of the closure)**.** *Let $V \subseteq X$ be a subspace of a Banach space X. Then ${}^\circ(V^\circ) = \overline{V}$.*

Proof. The space $^\circ(V^\circ)$ is the intersection of kernels of continuous linear operators and is therefore closed. The definitions imply that any $x \in V$ satisfies $x \in {}^\circ(V^\circ)$. Since $^\circ(V^\circ)$ is closed we therefore have $\overline{V} \subseteq {}^\circ(V^\circ)$. By the separation theorem, any $z \notin \overline{V}$ can be separated from $^\circ(V^\circ)$, i.e., there exists $F \in V^\circ$ with $F(z) \neq 0$, whence $z \notin {}^\circ(V^\circ)$. This shows the claimed equality of spaces. ∎

We recall that for Banach spaces X and Y and a continuous linear map $L : X \to Y$ the dual $L^* : Y^* \to X^*$ is defined by

$$L^*(F) = \langle F, L\,\cdot\rangle \in X^*.$$

We recall the closed range theorem. We denote by $\mathcal{L}(X,Y)$ the space of bounded and continuous maps from X to Y.

Theorem 1.5 (closed range theorem)**.** *Let $L \in \mathcal{L}(X,Y)$ be a continuous linear map between Banach spaces X and Y. The range $L(X)$ is closed in Y if and only if $L(X) = {}^\circ(\ker L^*)$.*

Proof. We have $f \in \ker L^*$ if and only if $\langle f, Lx\rangle = 0$ for all $x \in X$, which means $f \in L(X)^\circ$. We apply the foregoing lemma to the space $\ker L^* = L(X)^\circ$ and conclude the proof. ∎

The main application of the closed range theorem for our purposes is the characterization of solvability of operator equations. Recall that a Banach space Y is called reflexive if the map

$$J : Y \to Y^{**}, \quad Y \ni y \mapsto \langle \cdot, y\rangle$$

from Y to its bidual Y^{**} is an isomorphism.

Lemma 1.6 (Banach–Babuška–Nečas lemma)**.** *Let X be a Banach space and let Y be a reflexive Banach space. A linear map $L : X \to Y^*$ is an isomorphism if and only if the following three conditions are satisfied:*

(1) Continuity: $\|Lx\|_{Y^*} \leq C\|x\|_X$ for a constant $C > 0$ and all $x \in X$.

(2) There exists $\gamma > 0$ such that for all $x \in X$

$$\gamma\|x\|_X \leq \|Lx\|_{Y^*}.$$

(3) For every nonzero $y \in Y \setminus \{0\}$ there exists some $x \in X$ such that $\langle Lx, y\rangle \neq 0$.

Proof. Let conditions (1)–(3) be satisfied. Then, by (1), L is continuous and, by (2), it is injective because $Lx = 0$ implies $x = 0$. Hence, L is bijective as a map from X to its range $L(X)$. The inverse $L^{-1} : L(X) \to X$ is continuous because, by (2),

$$\|L^{-1}z\|_X \leq \gamma^{-1}\|LL^{-1}z\|_{Y^*} = \gamma^{-1}\|z\|_{Y^*}.$$

The continuity of L and L^{-1} implies that $L(X)$ is closed. The closed range theorem then teaches

$$L(X) = {}^\circ(\ker L^*) \subseteq Y^*. \tag{1.1}$$

Let us write down the polar set of $\ker L^* \subseteq Y^{**}$ explicitly:

$$^\circ(\ker L^*) := \{v \in Y^* : \langle u, v\rangle = 0 \text{ for all } u \in \ker L^*\}.$$

We furthermore observe from the definition of L^* that

$$u \in \ker L^* \iff \langle L^*u, x\rangle = 0 \text{ for all } x \in X \iff \langle u, Lx\rangle = 0 \text{ for all } x \in X.$$

Since Y is reflexive, we see that

$$u \in \ker L^* \iff \langle Lx, J^{-1}u \rangle = 0 \text{ for all } x \in X$$

for $J^{-1}u \in Y$ and the isomorphism J. Property (3) therefore implies that $J^{-1}u = 0$ and so $\ker L^* = \{0\}$. By (1.1), we then have that $L(X) = Y^*$. Thus, L is an isomorphism.

The proof of the converse direction is immediate and left as an exercise to the readers. ■

Condition (2) of Lemma 1.6 is often called the inf-sup condition because γ can be represented as

$$\gamma = \inf_{x \in X \setminus \{0\}} \sup_{y \in Y \setminus \{0\}} \frac{\langle Lx, y \rangle}{\|x\|_X \|y\|_Y}.$$

§2. Quasi-optimality of the Galerkin method

We consider the situation of a Banach space X and reflexive Banach space Y^*. Suppose and $x \in X$ and $f \in Y^*$ satisfy $Lx = f$. In any practical simulation we need to approximate the infinite-dimensional spaces X and Y. Suppose we are given closed subspaces $X_h \subseteq X$ and $Y_h \subseteq Y$ with the inclusion mappings ι_X and ι_Y. Then the *Galerkin method* is to find $x_h \in X_h$ such that Lx_h equals f when restricted to test functions of Y_h.

Theorem 1.7 (Galerkin method)**.** *Consider a Banach space X and reflexive Banach Y with closed subspaces $X_h \subseteq X$ and $Y_h \subseteq Y$ with $L \in \mathcal{L}(X, Y^*)$ and let $x \in X$ solve $Lx = f$ for some $f \in Y^*$. Assume that there exists $\gamma_h > 0$ such that*

$$\gamma_h \le \inf_{\xi_h \in X_h \setminus \{0\}} \frac{\|L\xi_h\|_{Y_h^*}}{\|\xi_h\|_X}$$

and that for any $y_h \in Y_h \setminus \{0\}$ there exists $\xi_h \in X_h$ with $\langle L\xi_h, y_h \rangle \neq 0$. Then there exists a unique solution $x_h \in X_h$ to $\iota_Y^* L x_h = \iota_Y^* f$. It satisfies the error bound

$$\|x - x_h\|_X \le \left(1 + \frac{\|\iota_Y^* L\|_{\mathcal{L}(X,Y^*)}}{\gamma_h}\right) \inf_{z_h \in X_h} \|x - z_h\|_X.$$

Proof. The existence and uniqueness of x_h follow from the Banach–Babuška–Nečas lemma. For any $z_h \in X_h$ we have that

$$\gamma_h \|z_h - x_h\|_X \le \|L(z_h - x_h)\|_{Y_h^*} = \sup_{y_h \in Y_h \setminus \{0\}} \frac{\langle L(z_h - x_h), y_h \rangle}{\|y_h\|_Y}. \tag{1.2}$$

Since $\langle Lx_h, y_h \rangle = \langle f, y_h \rangle$ from the solution property of x_h, we deduce from the continuity of L that

$$\gamma_h \|z_h - x_h\|_X \le \|\iota_Y^* L\|_{\mathcal{L}(X,Y^*)} \|\langle L(z_h - x), y_h \rangle\|.$$

The asserted bound follows from the triangle inequality $\|x - x_h\|_X \le \|x - z_h\|_X + \|z_h - x_h\|_X$ and the infimum over z_h. ■

The main application is that X_h and Y_h are finite-dimensional. Condition (1.2) is then called the discrete inf-sup condition. The nondegeneracy assumption means that the spaces have the same dimension. In practice, we think of h being a mesh parameter that increases the resolution by being decreased. The error bound for the Galerkin method is proportional to γ_h^{-1} and, therefor, it is important to have the inf-sup condition *uniformly in h*.

Example 1.8. In variationally formulated and coercive PDEs over a Hilbert space X we choose $X = Y$ and a bilinear form $a : X \times X \to \mathbb{R}$. It induces a linear operator $L : X \to X^*$ by $x \mapsto a(x, \cdot)$. Given $f \in X^*$, the equation $Lx = f$ then means

$$a(x, y) = \langle f, y\rangle \quad \text{for all } y \in X.$$

For $X_h \subseteq X$ as above, the discrete equation $\iota_Y^* L x_h = \iota_X^* f$ means

$$a(x_h, y_h) = \langle f, y_h\rangle \quad \text{for all } y_h \in X_h$$

and should be familiar to the reader from previous elementary courses. As the most important example we mention $X = H_0^1(\Omega)$ as the usual Sobolev space over a suitable domain Ω and the form a related to an elliptic second-order operator. The above theorem then resembles Céa's lemma.

§3. Saddle-point problems in reflexive spaces

Minimization of functionals subject to linear constraints can be re-formulated with Lagrange multipliers. The usual (formal) derivation of a necessary condition of

$$\text{minimize } \frac{1}{2}\langle Au, u\rangle - \langle f, v\rangle \quad \text{over } V \text{ subject to } Bu = 0$$

is to introduce a Lagrange multiplier $p \in M$ such that

$$Au + B'u = f$$

with an adjoint operator $B' := B^* J_M$. We want to study the well-posedness of such formulations. In this situation, we are given a product spaces $X = V \times M$ where the operator L has block structure. Given $F \in V^*$ and $G \in M^*$, a so-called *saddle-point problem* has the format

$$L \begin{bmatrix} u \\ p \end{bmatrix} := \begin{bmatrix} A & B' \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}. \tag{1.3}$$

The conditions of the Banach–Babuška–Nečas lemma can equivalently formulated as conditions on A and B. We are given a bounded linear B operator that is not surjective but has a bounded inverse on its range. In finite dimensions, these are the full rank rectangular matrices. We want to study analogous mapping properties in reflexive Banach spaces.

Lemma 1.9. *Let V and M be reflexive Banach spaces and $B \in \mathcal{L}(V, M^*)$, with $Z := \ker B$, satisfy*

$$0 < \beta = \inf_{\mu \in M\setminus\{0\}} \frac{\|B^* J_M \mu\|_{V^*}}{\|\mu\|_M}. \tag{1.4}$$

Then, $B^* J_M : M \to Z^\circ$ is an isomorphism with $\|(B^* J_M)^{-1}\|_{\mathcal{L}(Z^\circ, M)} \leq \beta^{-1}$.

Proof. We observe that the range of $B^* J_M$ is indeed a subset of Z° because $\langle B^* J_M \mu, z\rangle = \langle J\mu, Az\rangle = 0$ for any $\mu \in M$ and any $z \in Z$. By the above assumptions, $B^* J_M$ is continuous (property (1) of Lemma 1.6) and (1.4) implies that property (2) of Lemma 1.6 is satisfied. As in the proof of Lemma 1.6 we therefore see that $B^* J_M$ and its inverse are continuous. The closed range theorem then shows

$$B^* J_M(M) = {}^\circ(\ker((B^* J_M)^*)).$$

It is direct to verify

$$u \in \ker((B^* J_M)^*) \iff J_V^{-1} u \in Z.$$

Thus the range equals Z° and we have established the isomorphism. The bound on the norm is left as an exercise. ■

Brezzi's splitting theorem performs block elimination in the above system. It is our main criterion for saddle-point problems.

Theorem 1.10 (Brezzi splitting)**.** *Let V and M be reflexive Banach spaces and $B \in \mathcal{L}(V, M^*)$, with $Z := \ker B$. The operator*

$$L : X \to X^* \quad \textit{with } X = V \times M$$

from (1.3) *is an isomorphism if and only if A is an isomorphism from Z to Z^* with*

$$0 < \alpha = \inf_{z \in Z \setminus \{0\}} \frac{\|Az\|_{Z^*}}{\|z\|_X}$$

and B satisfies the inf-sup condition (1.4)*. Given $F \in V^*$ and $G \in M^*$, the unique solution $(u,p) \in V \times M$ to* (1.3) *block satisfies*

$$\begin{aligned}
\|u\|_V &\le \alpha^{-1} \|F\|_{V^*} + \beta^{-1}(1 + \frac{C_A}{\alpha}) \|G\|_{M^*},\\
\|p\|_M &\le \beta^{-1}(1 + \frac{C_A}{\alpha}) \|F\|_{V^*} + \beta^{-1}(1 + \frac{C_A}{\alpha}) \frac{C_A}{\beta} \|G\|_{M^*}.
\end{aligned}$$

Here, $C_A = \|A\|_{\mathcal{L}(V,V^*)}$.

Proof. We have seen in the previous lemma that $B^* J_M : M \to Z^\circ$ is an isomorphism, and so is $(B^* J_M)* : (Z^\circ)^* \to M^*$ with the same continuity constant for the inverse. Hence, for the given $G \in M^*$ there exists $\eta \in (Z^\circ)^*$ with $(B^* J_M)^* \eta = G$ with $\|\eta\|_{(Z^\circ)^*} \le \beta^{-1} \|G\|_{M^*}$. We denote by $\hat\eta \in V^{**}$ the Hahn–Banach extension of η that coincides with η on Z° and has the same norm. Then, the element $u_0 := J_V^{-1} \hat\eta$ satisfies for any $\mu \in M$ that

$$\langle B u_0, \mu \rangle = \langle J_M \mu, B u_0 \rangle = \langle B^* J_M \mu, u_0 \rangle = \langle B^* J_M \mu, J_V^{-1} \hat\eta \rangle = \langle \hat\eta, B^* J_M \mu \rangle = \langle (B^* J_M)^* \eta, \mu \rangle = \langle G, \mu \rangle.$$

Hence, $B u_0 = G$ with $\|u_0\|_V \le \beta^{-1} \|G\|_{M^*}$. Upon defining $w := u - u_0$, we reformulate the original problem into

$$\begin{aligned}
Aw + B'p &= F - A u_0\\
Bw &= 0.
\end{aligned}$$

We restrict the first equation to Z and obtain from the assumed isomorphism property of A that there exists a unique $w \in Z$ satisfying

$$\iota_Z^* A w = \iota_Z^* (F - A u_0)$$

with $\|w\|_V \le \alpha^{-1}(\|F\|_{V^*} + C_a \|u_0\|_V)$. Here, ι_Z is the inclusion of Z to V (this notation is short, but not consistent with the above one).

Since $F - A(u_0 + w) \in Z^\circ$, the foregoing lemma yields the existence of $p \in M$ with

$$B'p = F - A(u_0 + w)$$

and

$$\|p\|_M \le \beta^{-1}(\|F\|_{V^*} + C_a\|u_0 + w\|_V).$$

Hence, $u := u_0 + w$ and p solve the saddle-point problem. The asserted norm bounds follow from directly tracing the constants in the above estimates. The proof of the converse statement is left as an exercise. ■

Remark 1.11. The saddle-point problem is encountered more often in a variational form in the literature and reads as

$$\begin{aligned} a(u,v) + b(v,p) &= F(v) \quad \text{for all } v \in V \\ b(u,q) &= G(q) \quad \text{for all } q \in M \end{aligned}$$

for bounded bilinear forms $a : V \times V \to \mathbb{R}$ and $b : V \times M \to \mathbb{R}$. This is of course equivalent to the above formulation. Indeed, we see that $Au := a(u,\cdot) \in V^*$ and $Bu := b(u,\cdot) \in M^*$. It is easy to check that $B'\mu$ then equals $b(\cdot,\mu) \in V^*$ and that the kernel can be written as

$$Z = \{v \in V : \forall \mu \in M b(v,\mu) = 0\}.$$

♦

If we want to discretize the saddle-point problem with closed subspaces $V_h \subseteq V$ and $M_h \in\subseteq M$, we can derive well-posedness and an error bound by applying Theorem 1.7 to $X_h = V_h \times M_h$. We apply Theorem 1.10 to the discrete setting and see that the (global) discrete inf-sup condition follows from the conditions

$$0 < \alpha_h = \inf_{z_h \in Z_h \setminus \{0\}} \frac{\|Az_h\|_{Z_h^*}}{\|z_h\|_X} \quad \text{and} \quad 0 < \beta_h = \inf_{\mu_h \in M_h \setminus \{0\}} \frac{\|B'\mu_h\|_{V_h^*}}{\|\mu\|_{M_h}}. \tag{1.5}$$

Here, $Z_h := \ker(\iota_M^* B \iota_V)$, equivalently written as

$$Z_h = \{v_h \in V_h : \forall \mu_h \in M_h \ \langle Bv_h, \mu_h\rangle = 0\}$$

is the *discrete kernel*. It is **very important** to note that in general we must expect $Z_h \not\subseteq Z$. Usually, the condition on α_h is not very critical, for example when A is coercive. The condition on β_h is very delicate and is linked to the compatibility of the two discrete spaces. We note the following consequence.

Corollary 1.12. *Let the conditions of Theorem 1.10 hold and let the closed subspaces $V_h \subseteq V$ and $M_h \subseteq M$ satisfy* (1.5)*. Let $(u,p) \in V \times M$ solve the saddle-point problem with right-hand side (F,G). Then, there exists a unique pair $(u_h,p_h) \in V_h \times M_h$ solving*

$$\begin{aligned} \iota_V^* A u_h + \iota_V^* B' p_h &= \iota_V^* F \\ \iota_M^* B p_h &= \iota_M^* G. \end{aligned}$$

It satisfies

$$\|u - u_h\|_V + \|p - p_h\|_M \le C(\inf_{v_h \in V_h} \|u - v_h\|_V + \inf_{q_h \in M_h} \|p - q_h\|_M)$$

with a constant C that only depends on α_h, β_h, C_A.

2. Variational problems in H(div)

§1. Duality in Hilbert spaces

For a Hilbert space Y, the Riesz representation theorem establishes an isometry between Y and its dual. That is, we can identify any $y \in Y$ with the linear functional $\langle y, \cdot \rangle_Y$. For example, any element of $L^2(\Omega)^*$ can be represented by $\int_\Omega g \cdot \, dx$ for some $f \in L^2(\Omega)$. Such identifications are very common and culminate in statements like "Y is its own dual", but some care is necessary when working with them. In particular, when working with more than one Hilbert space, it must be clear with respect to which space we take this identification.

Definition 2.1 (Gelfand triplet)**.** Let X, Y be Hilbert spaces where X is densely embedded in Y. We know (Exercise A.5) that Y^* is then densely embedded in X^*. After identifying Y with Y^* we therefore have the chain of embeddings

$$X \to Y \to X^*.$$

This is called a *Gelfand triplet* and Y is called the *pivot space.* ♦

We proceed with the most prominent example in our lecture, which is related to Sobolev spaces.

Example 2.2 (embedding in negative Sobolev spaces)**.** Given a bounded polyhedral Lipschitz domain Ω, recall the spaces $H^1(\Omega)$ $H^1_0(\Omega)$. As usual, we write $H^1_0(\Omega)^* = H^{-1}(\Omega)$. We know that the embedding $H^1(\Omega) \subseteq L^2(\Omega)$ is dense and, after identifying $L^2(\Omega)$ with its dual, we arrive at the inclusions

$$H^1(\Omega) \subseteq L^2(\Omega) \subseteq H^1(\Omega)^* \quad \text{and} \quad H^1_0(\Omega) \subseteq L^2(\Omega) \subseteq H^{-1}(\Omega).$$

Warning 2.3 (pivot space)**.** In stating such inclusions, it is of paramount importance to specify the pivot space. Anything else will be prone to heavy mistakes.

Example 2.4 (Dirichlet Laplacian)**.** We know the well-posedness of the weak Poisson equation $-\Delta u = f$ in Ω subject to the boundary condition $u|_{\partial\Omega} = 0$. The solution $u \in H^1_0(\Omega)$ is the Riesz representative of the functional $f \in H^{-1}(\Omega)$. If there exists $T_f \in L^2(\Omega)$ with $f = \int_\Omega T_f \cdot \, dx$, we use the identification of $L^2(\Omega)$ with itself to interpret the inclusion $f \in H^{-1}(\Omega)$ and say that "f is an L^2 function". Without specifying the underlying identification, the statement obviously makes no sense because the elements $H^{-1}(\Omega)$ are not functions. Note that not every element of $H^{-1}(\Omega)$ may have an L^2 representation.

Example 2.5 (Neumann Laplacian)**.** The Neumann Laplacian problem is to find $u \in H^1(\Omega)$ with $\int_\Omega u \, dx = 0$ and

$$-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad \partial u / \partial \nu = 0 \quad \text{on } \partial\Omega$$

for the outer unit normal ν. A necessary compatibility condition of f comes from the divergence theorem and reads as

$$\int_\Omega f \, dx = -\int_\Omega \Delta u \, dx = -\int_{\partial\Omega} \partial u / \partial \nu \, ds = 0.$$

We denote $Z := H^1(\Omega)/\mathbb{R} := \{v \in H^1(\Omega) : \int_\Omega v\,dx = 0\}$ and recall the weak formulation of the problem, namely: find $u \in Z$ such that

$$\int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f v\,dx.$$

This weak problem is, of course, well posed for any $f \in L^2(\Omega)$. The constraint on the average of f is not needed. The point is that any constant function f will result in $\int_\Omega f v\,dx = 0$ for any $v \in Z$ and therefore represents the zero element of Z^*. Indeed, the inclusion $Z \subseteq L^2(\Omega)$ is not dense. The correct pivot space in the Gelfand triple is therefore the space $L^2_0(\Omega)$ of L^2 functions with vanishing average. This resembles the above compatibility condition.

Example 2.6 (Neumann Laplacian as a saddle-point problem)**.** The Neumann Laplacian problem can be posed as a variational problem over $V := H^1(\Omega)$. We denote by $M \approx \mathbb{R}$ the space of constant functions and introduce the operator $B : V \to M$, $v \mapsto \int_\Omega v \cdot dx$. Denoting by A the gradient inner product, we see that A is coercive on Z (Poincaré's inequality) and that B trivially satisfies the inf-sup condition. Therefore, there exists a constant $p \in M$ such that

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v\,dx + \int_\Omega pv,\,dx &= \int_\Omega f v\,dx && \text{for all } v \in V \\ \int_\Omega uq\,dx &= 0 && \text{for all } q \in M. \end{aligned}$$

It is easy to see that $p = \fint_\Omega f\,dx$ equals the average of f, which conforms to the fact that only the projection of f to Z has an effect on u.

Given a bounded polyhedral Lipschitz domain Ω, we already know the spaces $H^1(\Omega)$ and $H^1_0(\Omega)$. The trace theorem teaches us that a function $v \in H^1(\Omega)$ admits boundary values $v|_{\partial\Omega} \in L^2(\partial\Omega)$ in the sense of traces. That is, there exists a linear and continuous operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ that coincides with the usual restriction to the boundary when applied to continuously differentiable functions. We recall that $H^1_0(\Omega)$ is the closure of $C^\infty_c(\Omega)$ unter the $H^1(\Omega)$ norm and can be characterized as the subspace of $H^1(\Omega)$ of functions with vanishing trace. The range of the trace operator is customarily denoted by

$$H^{1/2}(\partial\Omega) := T(H^1(\Omega)).$$

(The reason for this notation will become clear later in this lecture.) It is equipped with the minimal extension norm

$$\|g\|_{H^{1/2}(\partial\Omega)} := \inf_{v \in H^1(\Omega): Tv = g} \|v\|_{H^1(\Omega)}.$$

The minimal extension is the solution to an elliptic boundary value problem (see Exercise A.8 for a similar computation). We denote by $H^{-1/2}(\partial\Omega)$ the dual space of $H^{1/2}(\partial\Omega)$. The norm in that space is, as usual, defined as

$$\|q\|_{H^{-1/2}(\partial\Omega)} = \sup_{v \in H^{1/2}(\partial\Omega)} \frac{\langle q, v\rangle}{\|v\|_{H^{1/2}(\partial\Omega)}}.$$

We have the Gelfand triplet

$$H^{1/2}(\partial\Omega) \subseteq L^2(\partial\Omega) \subseteq H^{-1/2}(\partial\Omega),$$

for which we will later verify that the embedding is indeed dense.

If we now define by $H^{1/2}(\Gamma)$ for some $\Gamma \subseteq \partial\Omega$ the range of the trace operator restricted to Γ we are not working on a closed manifold anymore. Formal integration by parts with a function $v \in H^{1/2}(\Gamma)$ will cause boundary terms unless it admits an extension by zero to a function $\tilde{v}$ in $H^{1/2}(\partial\Omega)$. The space of such functions is defined as

$$\tilde{H}^{1/2}(\Gamma) := \{v \in H^{1/2}(\Gamma) : \tilde{v} \in H^{1/2}(\partial\Omega)\}.$$

We observe that in general $H^{-1/2}(\Gamma)$ and $(\tilde{H}^{1/2}(\Gamma))^*$ are different spaces. This is a delicate issue that we will discuss in more detail.

§2. The space H(div)

If we consider the Dirichlet problem $-\Delta u = f$ for the Laplacian with $f \in L^2(\Omega)$, we notice that $\sigma := \nabla u$ is an element of $[L^2(\Omega)]^n$. But we know more, namely

$$\int_\Omega \sigma \cdot \nabla v \, dx = -\int_\Omega f v \, dx \quad \text{for all } v \in C_c^\infty(\Omega).$$

That is, σ is in L^2 and possesses a *weak divergence* in L^2. The space of such vector fields is denoted by

$$H(\operatorname{div}, \Omega) := \left\{ \sigma \in [L^2(\Omega)]^n : \exists f \in L^2(\Omega) \; \forall v \in C_c^\infty(\Omega) \int_\Omega \sigma \cdot \nabla v \, dx = -\int_\Omega f v \, dx \right\}.$$

The weak divergence is then denoted by $\operatorname{div} \sigma = f$. The space is endowed with the norm

$$\|v\|_{H(\operatorname{div},\Omega)} := \sqrt{\|v\|_{L^2(\Omega)}^2 + \|\operatorname{div} v\|_{L^2(\Omega)}^2}.$$

One can show that $H(\operatorname{div}, \Omega)$ is the closure of the smooth vector fields (up to the boundary) with respect to the norm $\|\cdot\|_{H(\operatorname{div},\Omega)}$.

Of course, any vector field whose components all belong to $H^1(\Omega)$ automatically belong to $H(\operatorname{div}, \Omega)$. But $H(\operatorname{div})$ fields are more general. For example (see Exercise A.11), a piecewise polynomial vector field with respect to a triangulation need not be globally continuous to belong to that space. It suffices that it does not jump across any face in the direction normal to that face.

Functions from $H(\operatorname{div})$ have traces in a certain sense. Integration by parts shows (for sufficiently smooth functions) that

$$\int_{\partial\Omega} \varphi \tau \cdot \nu \, dx = \int_\Omega \varphi \operatorname{div} \tau \, dx + \int_\Omega \tau \cdot \nabla \varphi \, dx \le \|\tau\|_{H(\operatorname{div},\Omega)} \|\varphi\|_{H^1(\Omega)}.$$

This means that the normal trace, assigning $\tau \cdot \nu|_{\partial\Omega}$ to any τ, is a bounded linear functional on $H^{1/2}(\partial\Omega)$.

Example 2.7 (inhomogeneous Neumann problem)**.** Given $g \in H^{-1/2}(\partial\Omega)$, the weak form of the Neumann problem $-\Delta u + u = f$, $\partial u / \partial \nu = g$ seeks $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g v \, ds \quad \text{for all } v \in H^1(\Omega).$$

It is well posed and its unique solution u satisfies $\nabla u \cdot \nu = g$ on $\partial\Omega$ as an identity of elements in $H^{-1/2}(\partial\Omega)$. We therefore see that the normal trace is surjective onto that space.

Previously, we could easily restrict $H^{1/2}$ functions from $\partial\Omega$ to a subset $\Gamma \subseteq \partial\Omega$. This is not possible for elements from $H^{-1/2}(\partial\Omega)$. Indeed, by our above interpretation of the normal trace through integration by parts, we think of an identity

$$\int_\Gamma \varphi\tau \cdot \nu \, dx = \int_{\partial\Omega} \hat{\varphi}\tau \cdot \nu \, dx$$

where $\hat{\varphi}$ is the zero extension of φ. But that extension need not belong to $H^{1/2}(\partial\Omega)$. All subsequent computations from above then will make no sense any more.

Example 2.8. For $g \in H^{-1/2}(\partial\Omega)$ and a (generic) subset $\Gamma \subseteq \partial\Omega$, the integral $\int_\Gamma g \, ds$ is not well defined because the constant 1 over Γ is not in $H^{1/2}(\partial\Omega)$ when continued by 0. We will study this in more detail later, but for the moment we consider another example.

Example 2.9 (taken from §2.5.1 of [BBF13])**.** We know that in two dimensions the function $u(x) = \log(|\log(|x|)|)$ belongs to $H^1(\Omega)$ and so its trace belongs to $H^{1/2}(\partial\Omega)$. For simpler computations, we take Ω to be the quarter segment $\Omega = \{x_1 > 0, x_2 > 0, |x| \leq 1/\exp(1)\}$. The tangential derivative of u along $\partial\Omega$ then belongs to $H^{-1/2}(\partial\Omega)$ (this will be proven later in the lecture) and is denoted by g. By direct computation, we see that $\int_{\partial\Omega} g \, ds$ is finite, but $\int_\Gamma g \, ds$ for $\Gamma = \{x_2 = 0\} \cap \partial\Omega$ is not.

§3. Mixed finite elements for Poisson's equation

In Poisson's equation we introduce an additional vector variable σ and set

$$\sigma = \nabla u, \qquad -\operatorname{div} \sigma = f.$$

For the variable σ above we require $\sigma \in H(\operatorname{div}, \Omega)$ and

$$\int_\Omega \operatorname{div} \sigma \, v \, dx = -\int_\Omega f v \, dx \quad \text{for all } v \in L^2(\Omega).$$

The relation $\sigma = \nabla u$ and integration by parts reveal for any $\tau \in H(\operatorname{div}, \Omega)$ that

$$\int_\Omega \sigma \cdot \tau \, dx = -\int_\Omega \operatorname{div} \tau u \, dx + \int_{\partial\Omega} u\tau \cdot \nu \, ds.$$

Assuming a homogeneous Dirichlet boundary condition for u we $\sigma \in H(\operatorname{div}, \Omega)$ and $u \in L^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega \sigma \cdot \tau \, dx + \int_\Omega \operatorname{div} \tau u \, dx &= 0 && \text{for all } \tau \in H(\operatorname{div}, \Omega), \\ \int_\Omega \operatorname{div} \sigma \, v \, dx &= -\int_\Omega f v \, dx && \text{for all } v \in L^2(\Omega). \end{aligned}$$

In this way we have formulated Poisson's equation as a saddle-point problem. This formulation is referred to as *mixed formulation.* As an exercise it is shown that the system satisfies the properties from Brezzi's splitting theorem and is therefore well-posed. We remark that we have explicitly imposed the $H(\operatorname{div}, \Omega)$ regularity for the vector variable but now merely ask u to belong to $L^2(\Omega)$. The property that σ is the weak gradient of u is implicitly contained in the first row of the system.

We want to identify appropriate finite element spaces leading to a stable discretization of the mixed Laplacian. Since $L^2(\Omega)$ functions do not require any continuity, a reasonable choice is to discretize it by the subspace $P_0(\mathcal{T})$ of piecewise constant (possibly discontinuous) functions with respect to a regular simplicial triangulation $\mathcal{T}$. For piecewise polynomial discretizations of $H(\operatorname{div}, \Omega)$ we have seen in Problem A.11 that for each face of the triangulation the component of the piecewise polynomial vector field must be continuous in the normal direction with respect to the face. We thus will use the normal directions at the faces as the degrees of freedom. For simplicity, we restrict ourselves to two space dimensions but remark that an analogous reasoning works in any dimension. We begin with the construction on a single triangle. We set

$$RT_0(T) := \{v \in [L^2(T)]^2 : v(x) = \begin{pmatrix} a \\ b \end{pmatrix} + cx \text{ for } a, b, c \in \mathbb{R}\}.$$

The vector fields of $RT_0(T)$ belong to a subset of the vector fields that are affine in each component. Obviously $\dim RT_0(T) = 3$. For the standard P_1 finite element, the degrees of freedom were the point evaluations at the vertices and we worked with the nodal basis of hat functions. Since here we want to enforce continuity of the normal component across edges we seek a basis $(\psi_E)_{E \in \mathcal{E}(T)}$, where $\mathcal{E}(T)$ is the set of edges of T, such that

$$\fint_F \psi_E \cdot \nu_F \, dx = \begin{cases} 1 & \text{if } E = F \\ 0 & \text{else.} \end{cases} \tag{2.1}$$

Here, ν_F is the outer normal vector of T restricted to the edge F. This property is achieved by the following definition

$$\psi_{T,E}(x) := \frac{|E|}{2|T|}(x - P_E)$$

where P_E is the vertex of T opposite to E. The proof of (2.1) is left as an exercise.

Remark 2.10 (finite element in the sense of Ciarlet). In the foregoing discussion, we have seen that we can uniquely determine functions from a finite-dimensional space of functions over T by linear functionals that need not be point evaluations (as it would be the case for the usual Lagrange basis of polynomials). Following the reasoning of Ciarlet [Cia78], one can abstractly define a *finite element* as a triplet $(T, \mathcal{P}, \mathcal{L})$ consisting of a bounded Lipschitz domain T of $\mathbb{R}^n$ (the element domain), a finite-dimensional space $\mathcal{P}$ of functions over T (the *shape functions*), and a set $\mathcal{L}$ of linear functionals over $\mathcal{P}$ that forms a basis of P^* (the *node functionals*). It is an exercise to verify that $(T, RT_0(T), \{\fint_E \bullet \cdot \nu_T \, ds : E \in \mathcal{E}(T)\})$ is a finite element. ♦

Globally, we then define

$$RT_0(\mathcal{T}) := \{v \in H(\operatorname{div}, \Omega) : \forall T \in \mathcal{T} \; v|_T \in RT_0(T)\}.$$

This space is called the Raviart–Thomas finite element space. We have seen that it consists of all vector fields that are in $RT_0(T)$ for every triangle T and that are normal-continuous across each edge. Given any interior edge E, we fix a normal vector. For the two neighbouring triangles T_+ and T_- this vector then points inwards to one of them and outwards to the other one. We use the convention that

$$\nu_E = \nu_{T_+} \quad \text{and} \quad \nu_E = -\nu_{T_-}$$

that is, ν_E is the outward pointing normal to T_+. This is graphically illustrated in Figure 2.1. If E is a boundary edge, we define $T_- = \emptyset$.

The functions

$$\psi_E(x) = \begin{cases} \psi_{T_+,E}(x) & \text{if } x \in T_+ \\ -\psi_{T_-,E}(x) & \text{if } x \in T_- \\ 0 & \text{else} \end{cases}$$

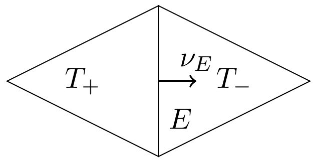


Figure 2.1.: Convention for the edge normal.

then form a global basis of $RT_0(\mathcal{T})$.

Lemma 2.11. *The functions $(\psi_E)_{E\in\mathcal{E}}$ form a basis of $RT_0(\mathcal{T})$. They satisfy $\fint_F \psi_E \cdot \nu_F \, dx = \delta_{EF}$.*

Proof. Exercise. ∎

The Raviart–Thomas space has a canonical interpolation operator, which reads for any sufficiently smooth vector field τ

$$I_{RT}\tau = \sum_{E\in\mathcal{E}} \fint_E \tau \cdot \nu_E \, ds \psi_E.$$

By construction, it satisfies the conservation property

$$\int_E I_{RT}\tau \cdot \nu_E \, ds = \int_E \tau \cdot \nu_E \, ds \quad \text{for any } E \in \mathcal{E}.$$

We will see that this operator is not well defined for functions in $H(\operatorname{div}, \Omega)$ but requires further regularity of τ. A sufficient criterion for $I_{RT}\tau$ to exist is for instance $\tau \in [H^1(\Omega)]^2$ because traces along edges are well defined due to the trace theorem. The following result shows H^1 stability of I_{RT}.

Theorem 2.12. *The Raviart–Thomas interpolation is stable with respect to the H^1 norm in the following sense. There exists a constant $C_{I_{RT}}$ only dependent on the shape regularity of $\mathcal{T}$ such that*

$$\|I_{RT}v\|_{H^1(\Omega)} \le C_{I_{RT}} \|v\|_{H^1(\Omega)} \quad \textit{for all } v \in [H^1(\Omega)]^2.$$

Proof. The restriction of $I_{RT}v$ to a triangle K can be written in terms of the basis expansion as follows

$$I_{RT}v|_K = \sum_{E\in\mathcal{E}(K)} \fint_E v \cdot \nu_E \, ds \psi_E.$$

A direct computation with the shape regularity shows for the basis function that $\|\psi_E\|_{L^2(K)} \lesssim h_K$. Similarly, $\|D\psi_E\|_{L^2(K)} \lesssim 1$. We recall the trace inequality and compute for the coefficient in front of ψ_E that

$$|\fint_E v \cdot \nu_E| \le |E|^{-1/2} \|v\|_{L^2(E)} \lesssim h_K^{-1} \|v\|_{L^2(K)} + \|Dv\|_{L^2(K)}.$$

We use the triangle inequality and compute

$$\|I_{RT}v\|_{L^2(K)} \le \sum_{E\in\mathcal{E}(K)} |\fint_E v \cdot \nu_E \, ds| \|\psi_E\|_{L^2(\Omega)} \lesssim \|v\|_{H^1(K)}.$$

In order to bound the gradient, we observe that $DI_{RT}v = DI_{RT}(v - \fint_K v\,dx)$ for the constant $\fint_K v\,dx$ (component-wise integral mean) because I_{RT} conserves constants (exercise). We then compute with trace and Poincaré inequalities that

$$
\begin{aligned}
\|DI_{RT}v\|_{L^2(K)} &= \|DI_{RT}(v - \fint_K v\,dx)\|_{L^2(\Omega)}\\
&\leq \sum_{E\in\mathcal{E}(K)} |\fint_E (v - \fint_K v\,dx)\cdot\nu_E\,ds|\|D\psi_E\|_{L^2(\Omega)}\\
&\lesssim h_K^{-1}\|(v - \fint_K v\,dx)\|_{L^2(K)} + \|\nabla v\|_{L^2(K)} \lesssim \|v\|_{H^1(K)}.
\end{aligned}
$$

Note that the constant in the Poincaré inequality scales like h_K. The claimed bound on the $H^1(\Omega)$ norm follows from using this local argument on each element domain. ■

The following so-called *commuting diagram property* is of particular importance. We denote by $\Pi_0 : L^2(\Omega) \to P_0(\mathcal{T})$ the L^2 projection on piecewise constants. It has the following representation (exercise)

$$
(\Pi_0 q)|_T = \fint_T q\,dx \quad \text{for all } q \in L^2(\Omega) \text{ and all } T \in \mathcal{T}.
$$

For vector variables, we use the same symbol Π_0 to denote the component-wise L^2 projection on $[P_0(\mathcal{T})]^2$.

Lemma 2.13 (commuting diagram property)**.** *The Raviart–Thomas interpolation* $I_{RT} : [H^1(\Omega)]^2 \to \mathbb{R}T_0(\mathcal{T})$ *satisfies*

$$
\operatorname{div} I_{RT}v = \Pi_0 \operatorname{div} v.
$$

In other words, the diagram

$$
\begin{tikzcd}
{[H^1(\Omega)]^2} \arrow[r, "\operatorname{div}"] \arrow[d, "I_{RT}"'] & L^2(\Omega) \arrow[d, "\Pi_0"] \\
RT_0(\mathcal{T}) \arrow[r, "\operatorname{div}"] & P_0(\mathcal{T})
\end{tikzcd}
$$

commutes.

Proof. Let $v \in [H^1(\Omega)]^2$. The divergence theorem shows for any $T \in \mathcal{T}$ with outer unit normal ν that

$$
\int_T \operatorname{div} I_{RT}v\,dx = \int_{\partial T} I_{RT}v\cdot\nu_T\,ds = \sum_{E\in\mathcal{E}(T)} \int_E I_{RT}v\cdot\nu|_E\,ds.
$$

For any edge $E \in \mathcal{E}(T)$, the operator I_{RT} conserves the integral of $v\cdot\nu|_E$. Thus

$$
\sum_{E\in\mathcal{E}(T)} \int_E I_{RT}v\cdot\nu|_E\,ds = \sum_{E\in\mathcal{E}(T)} \int_E v\cdot\nu|_E\,ds = \int_{\partial T} v\cdot\nu\,ds = \int_T \operatorname{div} v\,ds
$$

where we used again the divergence theorem. We combine the above two chains of identities and divide by the area of T to obtain

$$
\fint_T \operatorname{div} I_{RT}v\,dx = \fint_T \operatorname{div} v\,ds.
$$

The left integrals simply equals $\operatorname{div} I_{RT}v$ because the integrand is constant on T. The assertion follows with the above representation of Π_0 as the piecewise integral mean. ■

Let us now turn to the discretization of the mixed Laplacian. The mixed finite element approximation seeks $(\sigma_h, u_h) \in RT_0(\mathcal{T}) \times P_0(\mathcal{T})$ such that

$$\begin{aligned} \int_\Omega \sigma_h \cdot \tau_h \, dx - \int_\Omega \operatorname{div} \tau_h u_h \, dx &= 0 && \text{for all } \tau_h \in RT_0(\mathcal{T}), \\ \int_\Omega \operatorname{div} \sigma_h \, v_h \, dx &= -\int_\Omega f v_h \, dx && \text{for all } v_h \in P_0(\mathcal{T}). \end{aligned}$$

Theorem 2.14. *Given any $f \in L^2(\Omega)$, there is a unique solution $(\sigma_h, u_h) \in RT_0(\mathcal{T}) \times P_0(\mathcal{T})$ to the discrete mixed system. We have the error estimate*

$$\|\sigma - \sigma_h\|_{H(\operatorname{div},\Omega)} + \|u - u_h\|_{L^2(\Omega)} \le C\big(\inf_{\tau_h \in RT_0(\mathcal{T})} \|\sigma - \tau_h\|_{H(\operatorname{div},\Omega)} + \inf_{v_h \in P_0(\mathcal{T})} \|u - v_h\|_{L^2(\Omega)}\big)$$

for some constant C.

Proof. It suffices to prove the requirements from Brezzi's splitting theorem. The error estimate then follows from the abstract error estimate for the Galerkin method. For the proof of coercivity of the form

$$a(\sigma_h, \tau_h)$$

on the kernel Z_h, we first note that any $\tau_h \in Z_h$ satisfies by definition

$$\int_\Omega \operatorname{div} \tau_h v_h \, dx = 0 \quad \text{for all } v_h \in P_0(\mathcal{T}).$$

But since $\operatorname{div} \tau_h \in P_0(\mathcal{T})$, we see that $\operatorname{div} \tau_h = 0$ pointwise in Ω. Therefore

$$a(\tau_h, \tau_h) = \|\tau_h\|^2_{L^2(\Omega)} = \|\tau_h\|^2_{L^2(\Omega)} + \|\operatorname{div} \tau_h\|^2_{L^2(\Omega)} = \|\tau_h\|^2_{H(\operatorname{div},\Omega)},$$

which implies coercivity of a in $RT_0(\mathcal{T}) \subseteq H(\operatorname{div}, \Omega)$.

Let us prove the inf-sup condition for the form

$$b(\tau_h, v_h) := \int_\Omega \operatorname{div} \tau_h \, v_h \, dx.$$

Let any $v_h \in P_0(\mathcal{T})$ be given. In case that Ω is not convex, we increase the domain to a larger convex domain $\hat{\Omega}$ by adding suitable triangles, and we extend v_h by zero to a function $\hat{f} \in L^2(\hat{\Omega})$. On $\hat{\Omega}$ we then solve the weak form of the Dirichlet problem $\Delta \hat{w} = \hat{f}$ for some $\hat{w} \in H^1_0(\hat{\Omega})$. From the H^2 regularity on convex domains (Part I of this lecture) we deduce that $\hat{w} \in H^2(\hat{\Omega})$ with

$$\|\hat{w}\|_{H^2(\Omega)} \le C_{\mathrm{reg}} \|v_h\|_{L^2(\Omega)}, \quad \nabla \hat{w}|_\Omega \in [H^1(\Omega)]^2, \quad \text{and} \quad \operatorname{div} \nabla \hat{w} = v_h \text{ in } \Omega.$$

Since $\nabla \hat{w}$ in H^1, its Raviart–Thomas interpolation is well defined and satisfies, due to the commuting diagram property,

$$\operatorname{div} I_{RT} \nabla \hat{w} = \Pi_0 \operatorname{div} \nabla \hat{w} = \Pi_0(v_h) = v_h.$$

We furthermore have a bound on the $H(\operatorname{div}, \Omega)$ norm

$$\|I_{RT} \nabla \hat{w}\|^2_{H(\operatorname{div},\Omega)} = \|I_{RT} \nabla \hat{w}\|^2_{L^2(\Omega)} + \|v_h\|_{L^2(\Omega)} \lesssim \|\nabla \hat{w}\|^2_{H^1(\Omega)} + \|v_h\|^2_{L^2(\Omega)} \lesssim \|v_h\|^2_{L^2(\Omega)}.$$

We then compute

$$\sup_{\tau_h \in RT_0(\mathcal{T}) \setminus \{0\}} \frac{b(\tau, v_h)}{\|\tau\|_{H(\operatorname{div})} \|v_h\|_{L^2(\Omega)}} \ge \frac{b(I_{RT} \nabla \hat{w}, v_h)}{\|I_{RT} \nabla \hat{w}\|_{H(\operatorname{div})} \|v_h\|_{L^2(\Omega)}} = \frac{\|v_h\|^2_{L^2(\Omega)}}{\|I_{RT} \nabla \hat{w}\|_{H(\operatorname{div})} \|v_h\|_{L^2(\Omega)}} \gtrsim 1.$$

This proves the inf-sup condition with a constant that only depends on the shape regularity. ∎

Corollary 2.15. *If the solution to the Poisson equation satisfies $u \in H_0^1(\Omega) \cap H^2(\Omega)$, then*

$$\|\sigma - \sigma_h\|_{H(\mathrm{div},\Omega)} + \|u - u_h\|_{L^2(\Omega)} \leq h\|D^2 u\|_{L^2(\Omega)} + \|f - \Pi_0 f\|_{L^2(\Omega)}.$$

Proof. This follows from the interpolation error estimate and the piecewise Poincaré inequality. ■

§4. Selected aspects

Example 2.16 (mixed BVP)**.** Given a disjoint partition $\partial\Omega = \Gamma_D \cup \Gamma_N$ into a Dirichlet and a Neumann boundary, we consider the mixed boundary value problem $-\Delta u = f$ subject to $u|_{\Gamma_D} = u_D$ and $u|_{\Gamma_N} = 0$ for some given $u_D \in H^{1/2}(\Gamma_D)$. For simplicity we assume Γ_D to have positive surface measure. We introduce the space

$$H_N(\mathrm{div}, \Omega) := \{\tau \in H(\mathrm{div}, \Omega) : \tau \cdot \nu|_{\Gamma_N} = 0\}$$

and obtain the mixed formulation of the boundary value problem: Find $\sigma \in H_N(\mathrm{div}, \Omega)$ and $u \in L^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega \sigma \cdot \tau \, dx + \int_\Omega \mathrm{div}\, \tau u \, dx &= \langle \tau \cdot \nu, u_D \rangle \qquad \text{for all } \tau \in H_N(\mathrm{div}, \Omega), \\ \int_\Omega \mathrm{div}\, \sigma \, v \, dx &= -\int_\Omega f v \, dx \qquad \text{for all } v \in L^2(\Omega). \end{aligned}$$

Note that the Neumann condition enters as an essential boundary condition, while the Dirichlet condition is imposed weakly and appears on the right-hand side. This situation is "dual" to the usual formulation of the boundary value problem studied earlier. Inhomogeneous Neumann conditions have to be imposed in an essential way.

Transformation properties. When working with the Sobolev space $H^1(\Omega)$, for the usual Lagrange elements we know the affine equivalence to a reference element. We can parametrize T via an affine diffeomorphism $\Phi : \hat{T} \to T$ and know that the nodal functionals (point evaluations) are conserved under this transform. We also know the important relation $\nabla v = (D\Phi)^{-\top} \nabla \hat{v} \circ \Phi^{-1}$. In $H(\mathrm{div})$ problems, the situation is different because Φ does not map normal vectors to normal vectors and, thus, does not conserve the degrees of freedom. It turns out (and is well known from the theory of differential forms) that the right transform is the *pullback*, also known as *contravariant transform* or *Piola transform*. For an element $\hat{x} \in \hat{T}$ it acts on a vector field $\hat{q}$ as follows

$$x := \Phi(\hat{x}) \quad \text{and} \quad q(x) := |\det D\Phi(\hat{x})|^{-1} D\Phi(\hat{x}) \hat{q}(\hat{x}).$$

For affine Φ, the object $D\Phi$ can be thought of as a constant matrix, henceforth denoted by B. It is possible to verify that the normal vector ν to ∂T and the normal $\hat{\nu}$ to $\partial\hat{T}$ transform as

$$\nu(x) = \frac{1}{|B^{-\top} \hat{\nu}(\hat{x})|} B^{-\top} \hat{\nu}(\hat{x}),$$

see Exercise A.21. We furthermore have:

Lemma 2.17. *Let $q \in H(\mathrm{div}, T)$ be the Piola transform of $\hat{q}$ and $v \in H^1(T)$ be the affine transform of $\hat{v}$. Then*

$$\int_T \mathrm{div}\, q v \, dx = \int_{\hat{T}} \mathrm{div}\, \hat{q} \hat{v} \, dx, \quad \int_T q \cdot \nabla v \, dx = \int_{\hat{T}} \hat{q} \cdot \nabla \hat{v} \, dx, \quad \int_{\partial T} q \cdot \nu v \, ds = \int_{\partial \hat{T}} \hat{q} \cdot \hat{\nu} \hat{v} \, ds.$$

Proof. Exercise A.22. ■

§5. Error estimate in the H^{-1} norm

For the standard FEM, the Aubin–Nitsche trick can be used to establish an improved convergence rate for the L^2 norm of the error compared to the energy norm (the H^1 seminorm). For the mixed FEM, this is obviously impossible because it approximates u in L^2 with piecewise constants. This approximation will be of order h in case of full regularity, but not better. We first study the projected error $\Pi_h(u_h - u)$ and see that it exhibits a superconvergence phenomenon. For simplicity we state it on a convex domain where we know that the Poisson problem is H^2 regular.

Lemma 2.18. *Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded, convex polytope. Let $(\sigma, u) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$ solve the mixed system for the Poisson problem with right-hand side in $L^2(\Omega)$ and homogeneous Dirichlet boundary conditions. Let (σ_h, u_h) denote approximation from the discrete mixed system. Then the projected error satisfies*

$$\|\Pi_h(u_h - u)\|_{L^2(\Omega)} \lesssim h\|\sigma - \sigma_h\|_{H(\operatorname{div},\Omega)}.$$

Proof. Let $(\eta, w) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$ denote the solution to the mixed system with right-hand side $\Pi_h(u_h - u)$,

$$\begin{aligned} \int_\Omega \eta \cdot \tau \, dx + \int_\Omega \operatorname{div} \tau w \, dx &= 0 && \text{for all } \tau \in H(\operatorname{div}, \Omega), \\ \int_\Omega \operatorname{div} \eta \, v \, dx &= -\int_\Omega \Pi_h(u_h - u) v \, dx && \text{for all } v \in L^2(\Omega). \end{aligned}$$

We recall that $\eta \in H^1[(\Omega)]^n$ thanks to elliptic regularity. Thus, the interpolation I_{RT} is well defined. We test the second equation with $-(u_h - \Pi_h u)$ and obtain from the commuting diagram property of the interpolation I_{RT} and the solution properties of u and u_h that that

$$\|\Pi_h(u_h - u)\|^2_{L^2(\Omega)} = -\int_\Omega \operatorname{div} \eta \, \Pi_h(u_h - u) \, dx = -\int_\Omega \operatorname{div} I_{RT}\eta \, (u_h - u) \, dx = \int_\Omega (\sigma_h - \sigma) I_{RT}\eta \, dx.$$

We add and subtract η and use the first equation for η, which leads to

$$\int_\Omega (\sigma_h - \sigma) I_{RT}\eta \, dx = \int_\Omega (\sigma_h - \sigma)(I_{RT}\eta - \eta) \, dx - \int_\Omega \operatorname{div}(\sigma_h - \sigma) w \, dx.$$

The Galerkin equations for σ_h show that $\operatorname{div}(\sigma_h - \sigma)$ is L^2 orthogonal to the piecewise constant functions, so that we can subtract $\Pi_h w$ from w in the last integral. Thus, combining the previous two chains of identities with the Cauchy inequality yields

$$\|\Pi_h(u_h - u)\|^2_{L^2(\Omega)} \leq \|\sigma - \sigma_h\|_{L^2(\Omega)} \|\eta - I_{RT}\eta\|_{L^2(\Omega)} + \|\operatorname{div}(\sigma - \sigma_h)\|_{L^2(\Omega)} \|w - \Pi_h w\|_{L^2(\Omega)}.$$

Due to the H^2 regularity and $\eta = \nabla w$, we can use the error estimate for I_{RT} and the piecewise Poincaré inequality for $w - \Pi_h w$ and the elliptic regularity estimate to obtain

$$\|\eta - I_{RT}\eta\|_{L^2(\Omega)} + \|w - \Pi_h w\|_{L^2(\Omega)} \lesssim h\|w\|_{H^2(\Omega)} \lesssim h\|\Pi(u - u_h)\|_{L^2(\Omega)}.$$

The assertion follows from combining the foregoing two estimates. ∎

Corollary 2.19. *We have*

$$\|u - u_h\|_{H^{-1}(\Omega)} \lesssim h(\|\sigma - \sigma_h\|_{H(\operatorname{div},\Omega)} + \|u - u_h\|_{L^2(\Omega)}).$$

Proof. This follows from adding and subtracting $\Pi_h u$, the triangle inequality, direct computations with the H^{-1} norm and the projection Π_h, and the Poincaré inequality. ∎

§6. Estimates based on the hypercircle identity

The following result can be found in the literature under the name *Prager–Synge theorem* or *hypercircle identity.*

Lemma 2.20. *Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded Lipschitz domain and let $u \in H_0^1(\Omega)$ solve Poisson's problem with right-hand side $f \in L^2(\Omega)$. Then, any $\sigma \in H(\operatorname{div}, \Omega)$ with $-\operatorname{div}\sigma = f$ and any $v \in H_0^1(\Omega)$ satisfy the hypercircle identity*

$$\|\nabla(u - v)\|_{L^2(\Omega)}^2 + \|\nabla u - \sigma\|_{L^2(\Omega)}^2 = \|\nabla v - \sigma\|_{L^2(\Omega)}^2.$$

Proof. In the norm on the right-hand side we add and subtract ∇u and apply the binomial theorem to the squared norm. The result is the left-hand side plus the mixed expression

$$-2\int_\Omega \nabla(u - v) \cdot (\nabla u - \sigma)\, dx,$$

which equals zero as can be seen by integration by parts because $\operatorname{div}(\nabla u - \sigma) = 0$. ■

The fact that the right-hand side in the the hypercircle identity is independent of the unknown function u makes the result very useful for *a posteriori error estimation.* If we choose $v = u_h \in S_0^1(\mathcal{T})$ to be the Galerkin approximation to u with the standard finite element method, the hypercircle identity implies the error bound

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \le \|\nabla u_h - \sigma\|_{L^2(\Omega)} \quad \text{for any } \sigma \in H(\operatorname{div}, \Omega) \text{ with } -\operatorname{div}\sigma = f.$$

Once we make a choice for σ, the right-hand side is fully computable and is a guaranteed bound (there are no constants is the estimate) to the Galerkin error. Vice versa, if f is piecewise constant and $\sigma_h \in RT_0(\mathcal{T})$ is the discrete solution by the Raviart–Thomas method, we have the estimate

$$\|\nabla u - \sigma_h\|_{L^2(\Omega)} \le \|\nabla v - \sigma\|_{L^2(\Omega)} \quad \text{for any } v \in H_0^1(\Omega).$$

Such bounds are called *a posteriori* error estimates because they involve information of the discrete solution and thus are evaluated after the computation.

A direct consequence is:

Corollary 2.21. *Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded Lipschitz polytope triangulated by $\mathcal{T}$, let $f \in P_0(\mathcal{T})$ and let $u_h \in S_0^1(\mathcal{T})$ and $\sigma_h \in RT_0(\mathcal{T})$ be the approximations to $u \in H_0^1(\Omega)$ resp. ∇u by the standard resp. Raviart–Thomas FEM, where u solves Poisson's equation $-\Delta u = f$. We have the guaranteed a posteriori error bound*

$$\|\nabla(u - u_h)\|_{L^2(\Omega)}^2 + \|\nabla u - \sigma_h\|_{L^2(\Omega)}^2 = \|\nabla u_h - \sigma_h\|_{L^2(\Omega)}^2.$$

The following result states that the standard FEM and the Raviart–Thomas FEM are in some sense the optimal choice. We denote

$$Q_h(f) = \{\tau_h \in RT_0(\mathcal{T}) : -\operatorname{div}\tau_h = f\}.$$

Lemma 2.22. *Under the conditions of Corollary 2.21 we have*

$$\|\nabla u_h - \sigma_h\|_{L^2(\Omega)} = \min_{v_h \in S_0^1(\mathcal{T})} \min_{\tau_h \in Q_h(f)} \|\nabla v_h - \tau_h\|_{L^2(\Omega)}.$$

Proof. For any $\tau_h \in Q_h(f)$ and any $v_h \in S^1_0(\mathcal{T})$, the hypercircle reads

$$\|\nabla(u - v_h)\|^2_{L^2(\Omega)} + \|\nabla u - \tau_h\|^2_{L^2(\Omega)} = \|\nabla v_h - \tau_h\|_{L^2(\Omega)}.$$

Since u_h is the best approximation in the energy norm, the left-hand side is minimal for $v_h = u_h$, and therefore we have shown

$$\|\nabla u_h - \tau_h\|_{L^2(\Omega)} = \min_{v_h \in S^1_0(\mathcal{T})} \|\nabla v_h - \tau_h\|_{L^2(\Omega)}.$$

It remains to show that this expression is minimal for $\tau_h = \sigma_h$. To this end, we minimize the left-hand side over $Q_h(f)$, which is equivalent to

$$\frac{1}{2}\|\tau_h\|^2_{L^2(\Omega)} - \int_\Omega \nabla u_h \cdot \tau_h \, dx \to \min .$$

The Euler–Lagrange equation for the minimizer $\xi_h \in Q_h(f)$ of this quadratic minimization problem is (after integration by parts)

$$\int_\Omega \xi_h \cdot \tau_h \, dx = \int_\Omega \nabla u_h \cdot \tau_h \, dx = 0 \qquad \text{for all } \tau_h \in Q_h(0).$$

The inf-sup condition for the Raviart–Thomas method shows that there exists a Lagrange multiplier $w_h \in P_0(\mathcal{T})$ such that

$$\begin{aligned} \int_\Omega \xi_h \cdot \tau_h \, dx + \int_\Omega w_h \operatorname{div} \tau_h \, dx &= 0 && \text{for all } \tau_h \in RT_0(\mathcal{T}) \\ \int_\Omega \operatorname{div} \xi_h v_h \, dx &= -\int_\Omega f v_h \, dx && \text{for all } v_h \in P_0(\mathcal{T}). \end{aligned}$$

This shows that $\xi_h = \sigma_h$ is the solution to the Raviart–Thomas system. This establishes the asserted identity. ■

The foregoing result has shown that for bounding the error in the standard FEM, the optimal choice from $RT_0(\mathcal{T})$ for the upper bound is the result of the Raviart–Thomas FEM; and that the best choice from $S^1_0(\mathcal{T})$ for bounding the Raviart–Thomas error is the solution to the standard FEM. The lemma has shown that this choice is sharp in the sense that the upper bound is bounded by the errors of the two methods. The disadvantage is that, for example, for bounding the error of the standard FEM, an additional mixed linear system of more or less the same size needs to be solved, which is considered too expensive. Instead, a suitable $\tau_h \in Q_h(f)$ can be designed by a local construction. We restrict our attention to $n = 2$ for simplicity. We observe that ∇u_h is piecewise divergence-free but not globally in $H(\operatorname{div}, \Omega)$. The jump of a (possibly vector-valued) function v across an edge E is denoted by $[v]_E := v|_{T_+} - v|_{T_-}$ for the two elements $T_\pm$ sharing E. For boundary faces there is only one element T_+ and we set $[v]_E := v|_{T_+}$. In every element we have $\nabla u_h|_T \in RT_0(T)$. Once we have designed a piecewise RT_0 function τ_h^{pw} (not in $H(\operatorname{div}, \Omega)$ in general) with the property that $-\operatorname{div} \tau_h^{\mathrm{pw}}|_T = f|_T$ on every $T \in \mathcal{T}$ and $[\tau_h^{\mathrm{pw}}]_E \cdot \nu_E = -[\nabla u_h]_E \cdot \nu_E$ for every interior edge E, we have that $\tau_h^{\mathrm{pw}} = \tau_h - \nabla u_h$ for an element $\tau_h \in Q(f)$. It remains to evaluate the norm of τ_h^{pw}.

A possible construction is as follows. For a vertex z of the triangulation $\mathcal{T}$, we recall the vertex patch ω_z, which is the interior of the union of all triangles containing z. We define the set $\mathcal{E}(z)$ of edges containing z and denote by φ_z the corresponding $S^1(\mathcal{T})$ nodal basis function. We design a piecewise RT_0 function τ_h^z supported on ω_z as follows. For every edge $E \notin \mathcal{E}(z)$ we set the degree of freedom $\int_E \tau_z \cdot \nu_E \, ds = 0$. The remaining degrees of freedom are related to two faces per triangle. They are fixed by the conditions

$$\int_{\partial T} \tau_h^z \cdot \nu_T \, ds = -\int_T f\varphi_z \, dx \quad \text{for every } T \subseteq \overline{\omega}_z$$
$$[\tau_h^z]_E \cdot \nu_E = -\frac{1}{2}[\nabla u_h]_E \cdot \nu_E \quad \text{for every } E \in \mathcal{E}(z).$$

If z is an interior vertex, a simple degree-of-freedom count reveals that such choice can be achieved. If z is a boundary vertex, we enforce the jump condition only on the interior edges. Recall that for boundary faces there is no condition on the normal trace for a piecewise polynomial field to belong to $H(\operatorname{div}, \Omega)$. We then obtain as many conditions as degrees of freedom if we consider the connectivity components of $\partial\Omega$. A practical implementation is outlined in [Bra07, III§9].

Lemma 2.23. *The function $\tau_h^{\mathrm{pw}} := \sum_{z\in\mathcal{N}} \tau_h^z$ satisfies $-\operatorname{div} \tau_h^{\mathrm{pw}}|_T = f|_T$ on every $T \in \mathcal{T}$ and $[\tau_h^{\mathrm{pw}} \nu_E]_E \cdot = -[\nabla u_h]_E \cdot \nu_E$ for every interior edge E.*

Proof. Since the nodal basis functions φ_z form a partition of unity, the design of the functions τ_h^z implies that

$$\sum_{z\in\mathcal{N}(T)} \int_T \operatorname{div} \tau_h^z \, dx = \sum_{z\in\mathcal{N}(T)} \int_{\partial T} \tau_h^z \cdot \nu_T \, ds = \int_T f \, dx$$

and therefore $-\operatorname{div} \tau_h^{\mathrm{pw}}|_T = f|_T$ on every $T \in \mathcal{T}$. Furthermore, any edge E is shared by two vertices z_1, z_2, such that

$$[\tau_h^{\mathrm{pw}}]_E \cdot \nu_E = [\tau_h^{z_1}]_E \cdot \nu_E + [\tau_h^{z_2}]_E \cdot \nu_E = -[\nabla u_h]_E \cdot \nu_E.$$

■

We conclude the following reliability estimate:

Theorem 2.24. *Under the above assumptions (in particular f piecewise constant) we have the a posteriori error bound*

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \le \|\tau_h^{\mathrm{pw}}\|_{L^2(\Omega)}.$$

Algorithmic details on the implementation can be found in [Bra07, Chapter III §9].

Remark 2.25. One can prove that the bound is also efficient, that is the converse estimate holds up to a constant,

$$\|\tau_h^{\mathrm{pw}}\|_{L^2(\Omega)} \lesssim \|\nabla(u - u_h)\|_{L^2(\Omega)}.$$

♦

Remark 2.26. If $f \in L^2(\Omega)$ is not piecewise constant, we have

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \le \|\tau_h^{\mathrm{pw}}\|_{L^2(\Omega)} + \sqrt{\sum_{T\in\mathcal{T}} \frac{h_T^2}{\pi^2} \|f - {\textstyle\fint_T} f \, dx\|_{L^2(T)}^2},$$

see Exercise A.39. The additional term is referred to as *data oscillation.* ♦

3. Some details on Sobolev spaces and traces

We want to understand the origin of the notation $H^{1/2}(\partial\Omega)$ for the range of the trace operator and the connection to Sobolev scales.

§1. Sobolev spaces of non-integer order

We begin by defining the spaces H^s for non-integer values of s.

Definition 3.1 (Sobolev–Slobodeckij norm). Let $\Omega \subseteq \mathbb{R}^n$. For and $0 < s < 1$ and $v \in L^2(\Omega)$ we define

$$|v|_{H^s(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \, dxdy \right)^{1/2} \quad \in \mathbb{R} \cup \{\infty\}.$$

For a nonnegative integer $k \geq 0$ we define the space

$$H^{k+s}(\Omega) := \{v \in H^k(\Omega) : |\partial^\alpha v|_{H^s(\Omega)} < \infty \quad \text{for all multiindices with } |\alpha| = k\}$$

endowed with the *Sobolev–Slobodeckij norm*

$$\|v\|_{H^{k+s}(\Omega)} = \sqrt{\|v\|^2_{H^k(\Omega)} + \sum_{|\alpha|=k} |\partial^\alpha v|^2_{H^s(\Omega)}}.$$

♦

With this definition, we have a definition of the fractional-order space $H^{1/2}(\Omega)$. Since the boundary $\partial\Omega$ of our Lipschitz polytope Ω is a manifold, this does not directly give a definition of $H^{1/2}(\partial\Omega)$. In prior sections the latter space was already defined as the range of the trace operator, but for the moment we cancel that definition. The idea for defining $H^{1/2}(\partial\Omega)$ is to locally represent the boundary as the graph of a Lipschitz function, to flatten the boundary after localization with a suitable partition of unity, and to sum up the local $H^{1/2}$ norms of the transformed function. We recall the definition of a Lipschitz domain (first part of this lecture), where the open sets $U^1, \dots, U^N$ cover a neighbourhood of $\partial\Omega$ and, after rotating and shifting the coordinate system, $U^j \cap \partial\Omega = \{(z, \gamma_j(z)) : z \in \tilde{U^j}\}$ is the graph of a Lipschitz function γ_j with the domain on one side of the graph. Here, $\tilde{U^j} \subseteq \mathbb{R}^{n-1}$ is the domain of γ_j. We also consider a corresponding functions $\eta_j \in C_c^\infty(U^j)$ that form a partition of unity on the boundary, $\sum_j \eta_j = 1$ on $\partial\Omega$.

Definition 3.2 (H^s on the boundary). Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded Lipschitz domain. We say that $u : \partial\Omega \to \mathbb{R}$ belongs to $H^s(\partial\Omega)$ if each function $u_j = (\eta_j u)(\cdot, \gamma_j(\cdot))$ belongs to $H^s(\tilde{U^j})$. We define the (square of the) seminorm

$$|u|^2_{H^s(\partial\Omega)} := \sum_j \int_{\tilde{U}_j} \int_{\tilde{U}_j} \frac{|u_j(x) - u_j(y)|^2}{|x - y|^{n-1+2s}} dxdy.$$

(Note that x, y belong to $\mathbb{R}^{n-1}$). We define the norm $\|u\|_{H^s(\partial\Omega)} := (\|u\|^2_{L^2(\partial\Omega)} + |u|^2_{H^s(\partial\Omega)]})^{1/2}$. ♦

Remark 3.3. The value of the norm (but not its finiteness) in the above definition depends on the choice of the U^j and η_j. ♦

What we shall prove in this section is that the space $H^{1/2}(\partial\Omega)$ equals the range of the trace operator, i.e., every $g \in H^{1/2}(\partial\Omega)$ is the trace of some $u \in H^1(\Omega)$ with $\|u\|_{H^1(\Omega)} \le C\|g\|_{H^{1/2}(\partial\Omega)}$. Hence this alternative definition is equivalent to the one given above using the minimal extension norm.

We will not discuss traces of $H^s(\Omega)$ in detail, but what is important to observe is that functions from that space cannot have discontinuities on $(n-1)$-dimensional submanifolds if $s > 1/2$, but they can if $s < 1/2$, see Exercise A.24. The case $s = 1/2$ is critical and it turns out that such functions can only have certain discontinuities.

Example 3.4. A piecewise constant and discontinuous function u satisfies $u \notin H^{1/2}(\partial\Omega)$. But, for example, $u(t) = \log(|\log(|t|)|)$ belongs to $H^{1/2}(-1/\exp(1), 1/\exp(1))$. This will be proven later, cf. Exercise A.29.

Generally for $v \in L^2(\Omega)$, we denote by $\tilde{v} \in L^2(\mathbb{R}^n)$ the extension by 0.

Definition 3.5. Let $\Omega \subseteq \mathbb{R}^n$. We define

$$\widetilde{H}^{1/2}(\Omega) := \{v \in H^{1/2}(\Omega) : \tilde{v} \in H^{1/2}(\mathbb{R}^n)\}$$

with the norm

$$\|v\|_{\widetilde{H}^{1/2}(\Omega)} := \|\tilde{v}\|_{H^{1/2}(\mathbb{R}^n)}.$$

♦

We have $\|v\|_{H^{1/2}(\Omega)} \le \|v\|_{\widetilde{H}^{1/2}(\Omega)}$ for any $v \in \widetilde{H}^{1/2}(\Omega)$, see Exercise A.28.

Definition 3.6. We denote by $H^s_0(\Omega)$ the closure of $C^\infty_c(\Omega)$ with respect to the H^s norm. We denote by $H^{-s}(\Omega)$ the dual of $H^s_0(\Omega)$. ♦

Theorem 3.7 (density)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded Lipschitz domain and let $0 < s < 1$. Then, $H^s(\Omega)$ is a Banach space and we have $H^1(\Omega) \subseteq H^s(\Omega)$. The space $C^\infty(\overline{\Omega})$ is dense in $H^s(\Omega)$. We have*

$$H^s_0(\Omega) = \begin{cases} H^s(\Omega) & \text{if } 0 < s \le 1/2 \\ \widetilde{H}^s(\Omega) & \text{if } 1/2 < s < 1. \end{cases}$$

Proof. See for example [Gri85] or [Dob10]. ■

Remark 3.8. We stress the very important fact that $H^{1/2}(\Omega)$ is the closure of functions with compact support, but, at the same time, the elements in that space do not necessarily admit an $H^{1/2}$-regular extension by zero to the full space. ♦

§2. The range of the trace operator

Lemma 3.9 (trace)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded Lipschitz domain. The trace operator is continuous as a map from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$.*

Proof. For simplicity we assume $n = 2$. Let $u \in C^1(\bar{\Omega})$. We localize the boundary with the sets U^j and the cutoff functions η_j and use Exercise A.27. We can then assume without loss of generality that the support of u intersects the boundary such that $u|_{\partial\Omega}$ vanishes outside some $\Gamma \subseteq \partial\Omega$ which is the graph of a function γ_j over a subset of $\mathbb{R}^{n-1} = \mathbb{R}$, and $\Omega \subseteq \mathbb{R} \times \mathbb{R}_+$. We fix $x, y \in \mathbb{R}$ and define $\xi = (x-y)/2$ and $z = (\frac{1}{2}(x+y), |\xi|)$. We use the triangle inequality

$$|u(x,0) - u(y,0)| \leq |u(z) - u(x,0)| + |u(z) - u(y,0)|$$

and focus on the first term on the right-hand side. We use the fundamental theorem of calculus and obtain

$$|u(z) - u(x,0)| = |\int_0^1 \nabla u(x - t\xi, t|\xi|) \cdot \begin{pmatrix} -\xi \\ |\xi| \end{pmatrix} dt| \leq \sqrt{2}|\xi| \int_0^1 |\nabla u(x - t\xi, t|\xi|)|\, dt.$$

We square, divide by $|\xi|$ and integrate with respect to x and y. From symmetry in x and y we then obtain

$$|u(\cdot,0)|_{H^{1/2}(\Gamma)} \lesssim \int_0^1 \left(\int_\Gamma \int_\Gamma |\nabla u(x - t\xi, t|\xi|)|^2 \, dx\, dy \right)^{1/2} dt.$$

Here we have used Jensen's inequality $\int |f| \lesssim (\int f^2)^{1/2}$. Since u is compactly supported, we can replace the Γ in the integrals on the right-hand side by $\mathbb{R}$. We substitute with ξ

$$\int_\mathbb{R} \int_\mathbb{R} |\nabla u(x - t\xi, t|\xi|)|^2 \, dx\, dy = 2 \int_\mathbb{R} \int_\mathbb{R} |\nabla u(x - t\xi, t|\xi|)|^2 \, dx\, d\xi = 2 \int_\mathbb{R} \int_\mathbb{R} |\nabla u(x, t|\xi|)|^2 \, dx\, d\xi.$$

After changing coordinates $\xi \mapsto \xi/t$, we thus obtain

$$|u(\cdot,0)|_{H^{1/2}(\Gamma)} \lesssim \int_0^1 t^{-1/2} \|\nabla u\|_{L^2(\Omega)} \, dt \lesssim \|\nabla u\|_{L^2(\Omega)}.$$

This and density from Theorem 3.7 prove the continuity. ■

Conversely, any $v \in H^{1/2}(\partial\Omega)$ admits a bounded extension to $\hat{v} \in H^1(\Omega)$.

Lemma 3.10. *Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded Lipschitz domain. For every $v \in H^{1/2}(\partial\Omega)$ there exists an extension $\hat{v} \in H^1(\Omega)$ with $\|\hat{v}\|_{H^1(\Omega)} \leq C\|v\|_{H^{1/2}(\partial\Omega)}$ and $v = \hat{v}|_{\partial\Omega}$*

Proof. Again, we will prove this in the simplified situation of two dimensions to keep the technicalities to a minimum. The principal mathematical argument, however, is the same in higher dimensions. As in the previous proof we may assume that $v \in \widetilde{H}^{1/2}(\Gamma)$ for a bounded interval $\Gamma \subseteq \mathbb{R}$, and in view of Exercise A.25 we can assume that $\Gamma = \mathbb{R}$. We denote the coordinates of $\mathbb{R}^2$ by (x, y). We denote by ϕ the standard mollifier in $\mathbb{R}^2$ with support in the unit ball and unit integral, and set

$$\hat{v}(x,y) := \frac{1}{y} \int_\mathbb{R} \phi\left(\frac{z-x}{y}\right) v(z)\, dz, \qquad y > 0.$$

We compute (note that ϕ' has zero integral) the derivative and change coordinates,

$$\partial_x \hat{v}(x,y) = -\frac{1}{y^2} \int_\mathbb{R} \phi'\left(\frac{z-x}{y}\right) (v(z) - v(x))\, dz = \frac{1}{y} \int_{|z|<1} \phi'(z)(v(x) - v(x+yz))\, dz.$$

After squaring and integrating and observing that ϕ' is bounded, we compute

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}_+} |\partial_x \hat{v}(x,y)|^2\,dx\,dy &\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}_+} y^{-2}\int_{|z|<1} |v(x)-v(x+yz)|^2\,dz\,dx\,dy\\ &\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}_+} y^{-3}\int_{|x-w|<y} |v(x)-v(w)|^2\,dw\,dx\,dy\\ &\lesssim \int_{\mathbb{R}}\int_{\mathbb{R}} |v(x)-v(w)|^2\int_{|x-w|}^{\infty} y^{-3}\,dy\,dx\,dw.\end{aligned}$$

The y-integral equals $2^{-1}|x-w|^{-2}$, and therefore we have shown $\|\partial_x\hat{v}\|_{L^2(\Omega)} \lesssim |v|_{H^{1/2}(\Gamma)}$.

We next bound the derivative $\partial_y\hat{v}$. We observe that integration by parts implies

$$\int_{\mathbb{R}} \phi\left(\frac{z-x}{y}\right) dz + \int_{\mathbb{R}} \phi'\left(\frac{z-x}{y}\right)\frac{z-x}{y} dz = 0.$$

We can therefore compute

$$\partial_y\hat{v}(x,y) = -y^{-2}\int_{\mathbb{R}} \phi\left(\frac{z-x}{y}\right)(v(z)-v(x))dz - y^{-2}\int_{\mathbb{R}} \phi'\left(\frac{z-x}{y}\right)\frac{z-x}{y}(v(z)-v(x))dz.$$

The integrals are bounded in a similar fashion as before. ■

The preceding two results show that both definitions of $H^{1/2}(\partial\Omega)$ given in these notes are equivalent, and so are their norms.

Lemma 3.11. *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded Lipschitz domain. The derivative ∂_{x_j} continuously maps $H^{1/2}(\Omega)$ to the dual space $[\widetilde{H}^{1/2}(\Omega)]^*$.*

Proof. For the ease of notation we consider $n = 1$ and denote with x, y the Cartesian coordinates of $\mathbb{R}^2$. We consider functions $v \in H^{1/2}(\Omega)$ (with some continuation to $H^{1/2}(\mathbb{R})$). and $w \in \widetilde{H}^{1/2}(\Omega)$, which admits a bounded extension by zero to an object of $H^{1/2}(\mathbb{R})$. From previous proofs we know that these functions have bounded extensions $\hat{v} \in H^1(\mathbb{R}\times\mathbb{R}_+)$ and $\hat{w} \in H^1(\mathbb{R}\times\mathbb{R}_+)$. We obtain from integration by parts that

$$\int_{\Omega} \partial_x\hat{v}(x,y)\hat{w}(x,y)\,dx = -\int_y^{\infty}\int_{\mathbb{R}^n} (\partial_x\hat{v}(x,s)\partial_y\hat{w}(x,s) - \partial_y\hat{v}(x,s)\partial_x\hat{w}(x,s))\,ds\,dx.$$

For $y \to 0$ we obtain that

$$\int_{\Omega} \partial_x v(x)w(x)\,dx \lesssim \|\nabla\hat{v}\|_{H^1(\mathbb{R}\times\mathbb{R}_+)}\|\nabla\hat{w}\|_{H^1(\mathbb{R}\times\mathbb{R}_+)} \lesssim \|v\|_{H^{1/2}(\Omega)}\|w\|_{\widetilde{H}^{1/2}(\Omega)}.$$

for any such pair of functions. ■

The previous result is sharp in the sense that the partial derivative does not map $H^{1/2}(\Omega)$ to $H^{-1/2}(\Omega)$ for a bounded Lipschitz domain Ω, see Exercise A.29. But we have that the tangential derivative maps $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, which was already used in Example 2.9, see also Exercise A.30.

4. Corner singularities in planar domains

§1. Setting

This section provides a brief introduction to the regularity theory of elliptic second-order boundary value problems in Lipschitz polygons ("polygons" for short), that is, in open and bounded domains whose boundary can locally be represented as the graph of a piecewise affine function. As a simplification of the general situation presented in [Gri92, Chapter 2], we consider the Dirichlet Laplacian as a model case,

$$-\Delta u = f \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \text{ on } \partial\Omega.$$

In a Hilbert space setting, this problem has a unique solution Sobolev space $H_0^1(\Omega)$ for any $f \in H^{-1}(\Omega)$. More precisely, the operator

$$-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$$

is an isomorphism. If we restrict our attention to right-hand sides f from $L^2(\Omega)$, the range of the solution operator $(-\Delta)^{-1}|_{L^2(\Omega)}$ is a subspace of $H_0^1(\Omega)$, and regularity theory tries to find characterizations of this subspace. It is known that such solutions enjoy H^2 regularity in the interior that can be extended up to the boundary provided the latter is sufficiently smooth, say it belongs to the class C^2. In this case, it can be shown by local flattening and reflection techniques [Eva10] that $u \in H_0^1(\Omega) \cap H^2(\Omega)$ whenever $f \in L^2(\Omega)$. In domains with corners (such as polygons) this result is not generally true.

Example 4.1. Let $\Omega := \{(r, \theta) : 0 < r < 1 \text{ and } 0 < \theta < 3\pi/2\}$ denote the sector domain (r and θ are the usual polar coordinates). Then

$$u(r, \theta) = r^{2/3} \sin(2\theta/3)$$

belongs to $H^1(\Omega)$, satisfies zero boundary conditions near $(0, 0)$, but does not belong to $H^2(\omega)$ for any open subdomain $\omega \subset \Omega$ such that $(0, 0) \in \bar{\omega}$. On the other hand we have that $\Delta u = 0$, which belongs to $L^2(\Omega)$.

It will turn out that functions as in this example will describe characteristic singularities near corners. We shall prove that the operator $-\Delta$ maps $H_0^1(\Omega) \cap H^2(\Omega)$ to a closed subspace of $L^2(\Omega)$, whose orthogonal complement N has a dimension related to the corners of the domain. If Ω has finitely many corners, then N is finite-dimensional. This is the main decomposition theorem. Moreover, this characterization makes it possible to precisely predict the regularity of the solution using fractional Sobolev spaces. In the above example, the solution satisfies the regularity

$$u \in H^{5/3-\delta}(\Omega) \quad \text{for any } \delta > 0.$$

We will study how regularity in the fractional-order Sobolev spaces $H^s(\Omega)$ for $0 < s < 1$ is related to the corners of the domain.

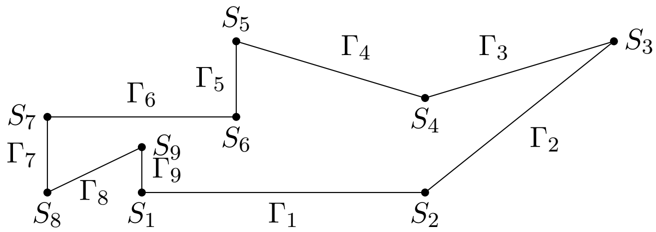


Figure 4.1.: Our notation for a polygon.

In what follows, $\Omega \subset \mathbb{R}^2$ is a bounded and open polygon with (for simplicity) finitely many corners. Thus, there exists a positive integer M such that the boundary consists of M many straight line segments $(\Gamma_j : j = 1, \dots, M)$ meeting at corners $(S_j : 1 = 1, \dots, M)$ where $S_j := \Gamma_j \cap \Gamma_{j+1}$, see Figure 4.1 for an illustration.

We consider the space

$$H(\Delta, \Omega) := \{v \in L^2(\Omega) : \Delta v \in L^2(\Omega)\}$$

with the norm

$$\|v\|_{H(\Delta,\Omega)} = \left(\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Note that first-order partial derivatives of functions from this space will in general only exist as distributions, but not as L^2 functions. Nevertheless, we can give a meaning to traces of functions from $H(\Delta, \Omega)$. We note that the outward pointing unit normal vector ν to $\partial\Omega$ exists almost everywhere on $\partial\Omega$ (namely in the interior of any of the segments Γ_j).

Lemma 4.2. *Consider the space*

$$W := H^2(\Omega) \cap H^1_0(\Omega) \subset H^2(\Omega).$$

The trace mapping $\gamma : H^2(\Omega) \to W^*$ defined by

$$v \mapsto \left[w \mapsto \int_{\partial\Omega} v \frac{\partial w}{\partial \nu}\, ds\right] =: \langle \gamma v, \cdot \rangle \in W^*$$

has a unique continuous extension to a linear map from $H(\Delta, \Omega)$ to W^*.

Proof. Let $v \in H^2(\Omega)$ and $w \in W$. Integration by parts (applied twice) shows

$$\int_\Omega v \Delta w\, dx = \int_{\partial\Omega} v \frac{\partial w}{\partial \nu}\, ds - \int_{\partial\Omega} w \frac{\partial v}{\partial \nu}\, ds + \int_\Omega w \Delta v\, dx.$$

Since w vanishes on the boundary, the second integral on the right-hand side equals zero. This and the Cauchy inequality establish

$$\begin{aligned} \int_{\partial\Omega} v \frac{\partial w}{\partial \nu}\, ds &= \int_\Omega v \Delta w\, dx - \int_\Omega w \Delta v\, dx \\ &\le \|v\|_{L^2(\Omega)} \|\Delta w\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} \|\Delta v\|_{L^2(\Omega)} \le C \|v\|_{H(\Delta,\Omega)} \|w\|_{H^2(\Omega)}. \end{aligned}$$

The result follows from density of $H^2(\Omega)$ in $H(\Delta, \Omega)$ (see Exercise A.35). ■

Remark 4.3. We interpret γu as a boundary trace for $u \in H(\Delta, \Omega)$ and write $u|_{\partial\Omega}$ instead of γu. ♦

§2. The decomposition theorems

Recall the notation $W := H^2(\Omega) \cap H_0^1(\Omega)$. We consider the Laplacian as an operator $\Delta : W \to L^2(\Omega)$. Injectivity and closed range property of Δ follow from Exercise A.32. We are interested in

$$N := \{v \in L^2(\Omega) : \forall w \in W \ (\Delta w, v)_{L^2(\Omega)} = 0\} = (\Delta W)^\perp.$$

These are the right-hand sides leading to singular solutions to the Laplacian.

Lemma 4.4. *We have $v \in N$ if and only if $v \in H(\Delta, \Omega)$ and*

$$\Delta v = 0 \text{ in } \Omega \qquad \text{and} \qquad v|_{\partial\Omega} = 0 \text{ in the sense of traces of } H(\Delta, \Omega).$$

Proof. The proof is left to the reader as an exercise. ■

Lemma 4.5. *Let $v \in N$ and let $U \subset \bar{\Omega}$ denote any neighbourhood of the corners $\{S_j\}$. Then $v \in C^\infty(\bar{\Omega} \setminus U)$.*

Proof. This is the classical interior regularity result, see [Eva10]. ■

Consider the corner number j with angle ω_j and the operator

$$\Lambda_j : H^2(0, \omega_j) \cap H_0^1(0, \omega_j) \to L^2(0, \omega_j)$$

defined by

$$\Lambda_j \varphi = -\varphi''.$$

We know from the spectral theory of self-adjoint compact operators that Λ_j has a discrete spectrum with nonnegative eigenvalues $\lambda_{j,m}^2$ $(m = 1, 2, 3, \dots)$. The corresponding L^2-normalized eigenfunctions are denoted by $\varphi_{j,m}$. It is well known that

$$\lambda_{j,m} = m\pi/\omega_j \quad \text{and} \quad \varphi_{j,m}(\theta) = \sqrt{2/\omega_j}\sin(\theta\lambda_{j,m}).$$

Given any corner S_j we denote the polar coordinates with origin S_j by (r_j, θ_j). We choose $\rho_j > 0$ small enough such that $D_{\rho_j} := \Omega \cap \{0 < r_j < \rho_j\}$ does not intersect with parts of $\partial\Omega$ other than $\Gamma_j \cup \Gamma_{j+1}$. We will sometimes use cut-off functions $\eta_j \in C^\infty(\bar{\Omega})$, $j = 1, \dots, M$ with mutually disjoint supports and the property

$$\eta_j = \begin{cases} 1 & \text{in an open neighbourhood of } S_j \\ 0 & \text{outside } D_{\rho_j}. \end{cases}.$$

We now fix one corner $S_j \equiv S$ and denote the polar coordinates with origin S by (r, θ). We write $\rho = \rho_j$ as well as $\lambda_m := \lambda_{j,m}$ and $\varphi_m := \varphi_{j,m}$, $\omega := \omega_j$.

The representation of the Laplacian in polar coordinates shows that any $v \in N$ satisfies

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0 \quad \text{for } 0 < \theta < \omega, 0 < r < \rho.$$

It can be shown that v has zero boundary conditions away from S_j (prove this as an exercise), see Lemma 4.5. For any $0 < r < \rho$, we have

$$v(r, \theta) \in H^2(0, \omega_j) \quad \text{(as a function of } \theta\text{)}$$

and thus

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{r^2}\Lambda_j v = 0 \quad \text{for } 0 < r < \rho. \tag{4.1}$$

Lemma 4.6. *Let $v \in C^\infty((0,\rho); H^2(0,\omega_j) \cap H_0^1(0,\omega_j))$ solve (4.1) and assume $v \in L^2(D_\rho)$. Then there exist real numbers α_m, β_m with*

$$|\alpha_m| \leq L m^{1/2} \rho^{-(\lambda_m+1)}$$

(L only dependent on v) such that

$$v(r,\theta) = \sum_{m\geq 1} \alpha_m r^{\lambda_m} \varphi_m(\theta) + \sum_{0<\lambda_m<1} \beta_m r^{-\lambda_m} \varphi_m(\theta).$$

Proof. The functions φ_m form a complete orthonormal system of $L^2(0,\omega)$. Thus

$$v(r,\theta) = \sum_{m\geq 1} v_m(r)\varphi_m(\theta) \quad \text{with the coefficient} \quad v_m(r) = \int_0^\omega v(r,\theta)\varphi_m(\theta)d\theta.$$

The differential equation implies

$$v_m''(r) + r^{-1}v_m'(r) - \lambda_m^2 r^{-2} v_m(r) = 0 \quad \text{for } 0 < r < \rho.$$

This ODE has the following solutions (Exercise A.33)

$$\begin{aligned} v_m(r) &= \alpha_m r^{\lambda_m} + \beta_m r^{-\lambda_m} \quad \text{for } \lambda_m > 0 \quad \text{(relevant here)} \\ v_m(r) &= \alpha_m + \beta_m \log(r) \quad \text{for } \lambda_m = 0 \quad \text{(not relevant here).} \end{aligned}$$

Squaring the coefficient relation, integrating, and using Cauchy's inequality implies

$$\int_0^\rho |v_m(r)|^2 r dr = \int_0^\rho |\int_0^\omega v(r,\theta)\varphi_m(\theta)d\theta|^2 r dr \leq \int_0^\rho \int_0^\omega |v(r,\theta)|^2 d\theta r dr = \|v\|^2_{L^2(D_\rho)} < \infty.$$

Thus in case $\lambda_m \geq 1$, we see that $\beta_m = 0$. Furthermore, if $\lambda_m \geq 1$, we see that

$$\frac{|\alpha_m|^2}{2\lambda_m + 2}\rho^{2\lambda_m+2} = |\alpha_m|^2 \int_0^\rho r^{2\lambda_m+1} dr \leq \|v\|^2_{L^2(D_\rho)}.$$

■

Theorem 4.7. *The dimension of N equals*

$$\sum_j \operatorname{card}\{\lambda_{j,m} : 0 < \lambda_{j,m} < 1\}.$$

Proof. Step 1. We begin by considering a fixed corner (number j) and the related eigenvalues λ_m and eigenfunctions φ_m. Let m be such that $\lambda_m \in (0,1)$. Recall the localization function $\eta \equiv \eta_j$ and the polar coordinates (r,θ) related to this corner. We define the function

$$u_m := \eta r^{-\lambda_m} \varphi_m(\theta).$$

We obviously have that $u_m \in H(\Delta,\Omega)$ (prove this as an exercise) with (generalized) zero boundary conditions.. We can thus solve for $v_m \in H_0^1(\Omega)$ with $\Delta v_m = \Delta u_m$ and set $\sigma_m := u_m - v_m$. We then have by construction that $\sigma_m \in H(\Delta,\Omega)$ and $\sigma_m|_{\partial\Omega} = 0$, furthermore $\Delta\sigma_m = 0$. By Lemma 4.5 we thus have $\sigma_m \in N$. Therefore we have shown that there exists $\sigma_m \in N$ such that

$$\sigma_m - \eta r^{-\lambda_m}\varphi_m(\theta) \in H^1(\Omega).$$

Step 2. Let $v \in N$. We have seen in the lemma that near our corner (number j) we have

$$v(r,\theta) - \sum_{m\geq 1} \alpha_m r^{\lambda_m}\varphi_m(\theta) - \sum_{0<\lambda_m<1} \beta_m r^{-\lambda_m}\varphi_m(\theta) = 0.$$

We have seen that $r^{-\lambda_m}\varphi_m(\theta)$ and σ_m only differ by an $H^1(\Omega)$ function, thus upon substituting we obtain

$$v(r,\theta) - \sum_{m\geq 1} \alpha_m r^{\lambda_m}\varphi_m(\theta) - \sum_{0<\lambda_m<1} \beta_m \sigma_m \in H^1(D_\rho).$$

It is proved as an exercise that (with the help of the bounds on α_m from Lemma 4.6)

$$\sum_{m\geq 1} \alpha_m r^{\lambda_m}\varphi_m(\theta) \in H^1(D_{\rho'}) \qquad \text{for any } 0 < \rho' < \rho.$$

Consequently, we infer that

$$v(r,\theta) - \sum_{0<\lambda_m<1} \beta_m \sigma_m \in H^1(D_{\rho'}).$$

Step 3. The interior regularity from Lemma 4.5 then shows that, in global notation, we have

$$w := v - \sum_j \sum_{0<\lambda_{j,m}<1} \beta_{j,m}\sigma_{j,m} \in H^1(\Omega).$$

On the other hand, since $w \in N \cap H^1(\Omega)$, we know by Lemma 4.5 that $w \in H^1(\Omega)$ is harmonic with zero boundary conditions. Thus, $w = 0$ and

$$v = \sum_j \sum_{0<\lambda_{j,m}<1} \beta_{j,m}\sigma_{j,m}.$$

■

For any corner S_j of the domain Ω we define the "singularity function" τ_j by

$$\tau_j(r_j,\theta_j) = \eta_j(r_j) r_j^{\lambda_{j,1}} \varphi_{j,1}(\theta_j).$$

These functions have the following properties.

Lemma 4.8. *The functions τ_j $(j = 1,\ldots,M)$ satisfy*

$$\tau_j \in H(\Delta,\Omega) \quad \textit{and} \quad \tau_j|_{\partial\Omega} = 0.$$

The functions $(\Delta\tau_j : j = 1,\ldots,M)$ are linearly independent. If $\lambda_{j,1} < 1$, then $\Delta\tau_j$ is not orthogonal to the space N.

Proof. Exercise A.37. ■

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^2$ be a connected and open polygonal domain and $f \in L^2(\Omega)$, and denote by $u \in H_0^1(\Omega)$ the solution to the Poisson equation*

$$-\Delta u = f \quad \textit{in } \Omega \quad \textit{and} \quad u = 0 \quad \textit{on } \partial\Omega.$$

Then there exist real coefficients $(c_1,\ldots,c_M) \in \mathbb{R}^M$ such that

$$u - \sum_{\substack{j \text{ with} \\ \omega_j > \pi}} c_j \tau_j \in H^2(\Omega).$$

Proof. We know that $\lambda_{j,m} = m\pi/\omega_j$. Thus

$$\lambda_{j,m} < 1 \quad \text{if and only if} \quad \omega_j > \pi \text{ and } m = 1.$$

Theorem 4.7 thus teaches us that

$$\dim N = \{j : \omega_j > \pi\}.$$

The functions τ_j for $\omega_j > \pi$ are linearly independent and thus, by a dimension argument, form a basis of N. Consequently, the space $L^2(\Omega)$ is spanned by the range of $\Delta((H_0^1(\Omega) \cap H^2(\Omega))$ and the functions $\Delta\tau_j$. Thus, given $f \in L^2(\Omega)$, there exists $w \in H_0^1(\Omega) \cap H^2(\Omega)$ and coefficients c_j such that

$$f = \Delta w + \sum_{\substack{j \text{ with} \\ \omega_j > \pi}} c_j \Delta\tau_j.$$

The assertion of the theorem follows from the uniqueness of the solution to the variational problem (i.e., apply Δ^{-1} on both sides). ■

We end this section with a quantification of regularity in Sobolev spaces of fractional order.

Theorem 4.10. *Let $\Omega \subset \mathbb{R}^2$ be a connected and open polygonal domain and $f \in L^2(\Omega)$. The solution $u \in H_0^1(\Omega)$ to the Poisson equation*

$$-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega$$

satisfies

$$u \in H^{1+s}(\Omega) \quad \text{for any } s < \min\{1, \min_{j=1,\dots,M} \frac{\pi}{\omega_j}\}.$$

Proof. Details are worked out in Exercise A.36. ■

5. Nonconforming FEM

§1. The Crouzeix–Raviart element

For standard methods we assumed the conformity property $V_h \subseteq V$, which led to a convenient error analysis via Céa's lemma. The idea of nonconforming methods is to gain more flexibility (in whatever sense) of the discretization by giving up that constraint. In general we will therefore work with discrete space $V_h \not\subseteq V$. We start with the Crouzeix–Raviart element as a basic example. For simplicity we shall work in $\mathbb{R}^2$. As usual, $P_1(\mathcal{T})$ is the space of piecewise affine (but possibly discontinuous) functions. Given a triangulation $\mathcal{T}$ of our usual bounded, open, polygonal Lipschitz domain Ω, we define

$$CR^1(\mathcal{T}) := \{v \in P_1(\mathcal{T}) : v \text{ is continuous is the midpoints of interior faces}\}.$$

The version with homogeneous boundary conditions reads

$$CR^1_0(\mathcal{T}) := \{v \in CR^1(\mathcal{T}) : v \text{ vanishes in the midpoints of boundary faces}\}.$$

We want to use this space to approximate the Dirichlet problem for the Laplacian, but we have the obvious difficulty that $CR^1_0(\mathcal{T})$ is not a subspace of $H^1_0(\Omega)$. For piecewise regular objects such as $v_h \in CR^1(\mathcal{T})$, we can evaluate a piecewise gradient

$$\nabla_h v_h \in L^2(\Omega) \text{ defined by } (\nabla_h v_h)|_T = \nabla(v_h|_T) \text{ for any } T \in \mathcal{T}$$

and define

$$|\!|\!| v |\!|\!|_h := \|\nabla_h v\|_{L^2(\Omega)} \quad \text{for any piecewise } H^1\text{-regular function.}$$

We can show:

Lemma 5.1. *The seminorm $|\!|\!| v |\!|\!|_h$ is a norm on the sum space $H^1_0(\Omega) + CR^1_0(\mathcal{T})$.*

Proof. Exercise A.43. ∎

The seminorm is induced by the bilinear form

$$a_h(v, w) := \int_\Omega \nabla_h v \cdot \nabla_h w \, dx \quad \text{for any } v, w \in H^1(\Omega) + CR^1(\mathcal{T}).$$

We have shown that a_h is an inner product on $CR^1_0(\mathcal{T})$, from which it is clear that, given $f \in L^2(\Omega)$, there exists a unique solution $u_h \in CR^1_0(\mathcal{T})$ to

$$a_h(u_h, v_h) = \int_\Omega f v_h \, dx \quad \text{for all } v_h \in CR^1_0(\mathcal{T}).$$

This is the the Crouzeix–Raviart (or nonconforming P_1) method for the Dirichlet problem of the Laplacian. For the implementation, we use the face-oriented basis functions with the property

$$⨍_F \psi_E \, ds = \delta_{E,F}$$

for interior faces E, F. We note that for piecewise affine functions stating that a function is continuous in a face midpoint is equivalent with the property that the average $\fint_E \cdot\, ds$ coincides on both neighbouring elements T_+ and T_-. On an element T with barycentric coordinates ϕ_1, ϕ_2, ϕ_3, and faces E_1, E_2, E_3 we use the convention that $\phi_j|_{E_j} = 0$, that is E_j is opposite to the vertex z_j. The local basis function ψ_{E_j} then reads

$$\psi_{E_j} = 1 - 2\varphi_j.$$

It is direct to verify that therefore the local stiffness matrix equals four times the local stiffness matrix of the standard FEM.

The nonconforming interpolation operator is defined via

$$I_h v := \sum_{E\in\mathcal{E}} \fint_E v\, ds \psi_E \quad \text{for any } v \in H^1(\Omega) + CR^1(\mathcal{T}).$$

It has the following important property.

Lemma 5.2 (projection property)**.** *The nonconforming interpolation satisfies for any $v \in H^1(\Omega)$*

$$\nabla_h I_h v = \Pi_0 \nabla v.$$

That is, the piecewise gradient of the interpolated function equals the best approximation of the gradient by piecewise constants.

Proof. Exercise A.42. ∎

We proceed with a basic error estimate.

Theorem 5.3. *Let Ω be an open and connected polygonal Lipschitz domain and assume that the solution u to the Poisson problem with $f \in L^2(\Omega)$ satisfies $u \in H^1_0(\Omega) \cap H^2(\Omega)$. Then*

$$\|\!|u - u_h|\!\|_h \lesssim h\|D^2 u\|_{L^2(\Omega)}.$$

Proof. We write $w_h := I_h u - u_h$ and use the triangle inequality

$$\|\!|u - u_h|\!\|_h \leq \|\!|u - I_h u|\!\|_h + \|\!|w_h|\!\|_h$$

and observe that the square of the second term on the right-hand side satisfies

$$\|\!|I_h u - u_h|\!\|_h^2 = a_h(I_h u - u_h, w_h) = a_h(I_h u, w_h) - \int_\Omega f w_h\, dx$$

because w_h belongs to the finite element space. We use the projection property of I_h and integration by parts for the term including $I_h u$ and compute

$$a_h(I_h u, w_h) = a_h(u, w_h) = \int_\Omega f w_h\, dx - \sum_{E\in\mathcal{E}} \int_E \nabla u \cdot \nu_E [w_h]_E\, ds$$

where $[\cdot]_E$ denotes as usual the jump across E (for boundary faces, we define it as the usual trace) and where we have used that $\nabla u \cdot \nu_E$ does not jump; indeed ∇u is H^1 regular. On any interior face E, the jump $[w_h]_E$ has vanishing average and is thus orthogonal to any constant function. We compute

$$\int_E \nabla u \cdot \nu_E [w_h]_E\, ds = \int_E \nabla_h(u - I_h u) \cdot \nu_E [w_h]_E\, ds \leq \|\nabla_h(u - I_h u)|_T\|_{L^2(E)} \|[w_h - \fint_E w_h\, ds]\|_{L^2(E)}.$$

With triangle, trace, and Poincaré inequalities as well as Exercise A.44, we deduce

$$\int_E \nabla u \cdot \nu_E [w_h]_E \, ds \lesssim h \|D^2 u\|_{L^2(T_+ \cup T_-)} \|\nabla_h w_h\|_{L^2(T_+ \cup T_-)}.$$

Altogether, we conclude the stated result from the finite overlap of face patches and the combination with the above arguments. ■

Remark 5.4. In the previous proof we could not use Céa's lemma. Instead, we directly worked with the H^2 regularity of the solution. This assumption can be relaxed with a more elaborate proof. ♦

§2. Application to the Stokes equations

We recall the Stokes equations. Given $f \in L^2(\Omega)$, we seek $u \in [H_0^1(\Omega)]^2$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } [H^{-1}(\Omega)]^2 \\ \operatorname{div} u &= 0 \quad \text{in } L_0^2(\Omega). \end{aligned}$$

Here, u is a vector field and Δ is defined component-wise. As usual, $L_0^2(\Omega)$ are the L^2 functions with vanishing integral over Ω. Since $\int_\Omega \operatorname{div} u \, dx = 0$ due to integration by parts, the second equation is indeed valid pointwise almost everywhere. The problem can be put in a saddle-point formulation. We set $V = [H_0^1(\Omega)]^2$, $M := L_0^2(\Omega)$ and

$$a(v,w) = \int_\Omega Dv : Dw \, dx, \quad b(v,q) = -\int_\Omega q \operatorname{div} v \, dx, \quad F(v) = \int_\Omega f \cdot v \, dx, \quad G = 0$$

and see that the above equation is equivalent to the usual saddle-point problem with this specific choices. The problem admits a unique solution. The proof obviously requires an inf-sup condition for the form b. We quote the result, which we will not prove in this lecture.

Theorem 5.5. *Given an open, bounded, connected Lipschitz domain Ω, there exists β such that*

$$0 < \beta = \inf_{q \in L_0^2(\Omega) \setminus \{0\}} \sup_{v \in [H_0^1(\Omega)]^2 \setminus \{0\}} \frac{\int_\Omega q \operatorname{div} v \, dx}{\|Dv\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}}$$

for some β.

We denote by Z the subspace of V of divergence-free vector fields

$$Z := \{v \in V : \operatorname{div} v = 0\}.$$

The solution u from the Stokes equations belongs to Z satisfies

$$a(u,v) = F(v) \quad \text{for all } v \in Z.$$

It is known from previous lectures that the design of Galerkin methods in Z is very difficult, see Exercise A.48. Discretizing the saddle–point problem is easier, but the resulting approximation will not be pointwise divergence-free in general. The advantage of a nonconforming discretization is that the discrete velocity field u_h is piecewise divergence-free, at the expense of the nonconformity $u_h \notin V$. We denote $V_h = [CR_0^1(\mathcal{T})]^2$, $M_h := P_0(\mathcal{T}) \cap L_0^2(\Omega)$ and

$$a_h(v,w) = \int_\Omega D_h v : D_h w \, dx, \quad b_h(v,q) = -\int_\Omega q \operatorname{div}_h v \, dx.$$

The nonconforming method seeks $u_h \in V_h$ and $p_h \in M_h$ such that

$$\begin{aligned} a_h(u_h, v_h) + b_h(v_h, p_h) &= F(v_h) && \text{for all } v_h \in V_h \\ b_h(u_h, q_h) &= 0 && \text{for all } q_h \in M_h. \end{aligned}$$

Lemma 5.6. *The discrete Stokes system has a unique solution* (u_h, p_h).

Proof. It suffices to check the discrete inf-sup condition. Given $q_h \in M_h$, the continuous inf-sup condition and the projection property of I_h show that

$$0 < \beta = \sup_{v \in [H_0^1(\Omega)]^2 \setminus \{0\}} \frac{\int_\Omega q_h \operatorname{div} v \, dx}{\|Dv\|_{L^2(\Omega)} \|q_h\|_{L^2(\Omega)}} \sup_{v \in [H_0^1(\Omega)]^2 \setminus \{0\}} \frac{\int_\Omega q_h \operatorname{div}_h I_h v \, dx}{\|Dv\|_{L^2(\Omega)} \|q_h\|_{L^2(\Omega)}}.$$

The projection property furthermore implies $\|D_h I_h v\|_{L^2(\Omega)} \leq \|Dv\|_{L^2(\Omega)}$. This implies the discrete inf-sup condition. ∎

It is not difficult to see that a solution u_h will satisfy $\operatorname{div}_h u_h = 0$. We consider $Z_h := \{v_h \in V_h : \operatorname{div}_h v_h = 0\}$.

Lemma 5.7. *The discrete solution u_h to the nonconforming Stokes discretization satisfies $u_h \in Z_h$ and*

$$a_h(u_h, v_h) = F(v_h) \quad \textit{for all } v_h \in Z_h.$$

Proof. This follows from testing with elements from Z_h. ∎

It is not difficult to obtain a basic a priori error estimate.

Theorem 5.8. *Assume the solution pair (u, p) to the Stokes system with $f \in L^2(\Omega)$ satisfies $u \in [H_0^1(\Omega)] \cap [H^2(\Omega)]^2$ and $p \in L_0^2(\Omega) \cap H^1(\Omega)$. Then, the error of the nonconforming FEM discretization satisfies*

$$|||u - u_h|||_h + \|p - p_h\|_{L^2(\Omega)} \lesssim h(\|D^2 u\|_{L^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)}).$$

Remark 5.9. These regularity assumptions are satisfied on convex domains (PDE literature). ♦

Proof of Theorem 5.8. It suffices to bound the norms of the errors $I_h u - u_h$ and $\Pi_0 p - p_h$ (use the triangle inequality and known bounds). The discrete inf-sup condition states

$$\begin{aligned} &|||I_h u - u_h|||_h + \|\Pi_0 p - p_h\|_{L^2(\Omega)} \\ &\quad \lesssim \sup_{\substack{|||w_h|||_h = 1 \\ \|q_h\|_{L^2(\Omega)} = 1}} \left[a_h(I_h u - u_h, w_h) + b_h(w_h, \Pi_0 p - p_h) + b_h(I_h u - u_h, q_h) \right]. \end{aligned}$$

The projection properties of I_h and Π_0 and the constraint on the divergence show that the last term on the right-hand side equals zero and that

$$a_h(I_h u - u_h, w_h) + b_h(w_h, \Pi_0 p - p_h) = a_h(u, w_h) + b_h(w_h, p) - \int_\Omega f \cdot w_h \, dx.$$

We proceed in a similar fashion as in the convergence proof for the Poisson equation. From piecewise integration by parts we obtain

$$a_h(u, w_h) + b_h(w_h, p) = \int_\Omega f w_h \, dx - \sum_{E \in \mathcal{E}} \int_E ((Du - pI_{2\times 2})\nu_E) \cdot [w_h]_E \, ds.$$

The conclusion of the proof is similar as in the Poisson case and left as an exercise. ∎

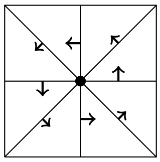


Figure 5.1.: Orientation of the normal vectors $\hat\nu_E$ around the vertex z

We have seen that the nonconforming method directly produces piecewise divergence-free solutions. It is possible to design a local basis of Z_h in an explicit construction:

- For each interior edge E we take a function $\alpha_E \in CR^1_0(\mathcal{T})$ such that

$$\fint_E \alpha_E \cdot \nu_E \, ds = 0, \quad \fint_E \alpha_E \cdot t_E \, ds = 1, \quad \fint_F \alpha_E \, ds = 0 \text{ for } F \neq E.$$

  Here $t_E = (-\nu_{E,2}, \nu_{E,1})$ is a unit tangent vector.

- For each interior vertex z with set of edges $\mathcal{E}(z)$ containing z we define $\alpha_z \in CR^1_0(\mathcal{T})$ as follows. All tangential components are set to zero. Also, the normal components are set to zero on those edges that do not touch z,

$$\fint_E \alpha_z \cdot t_E \, ds = 0 \quad \text{for all } E \in \mathcal{E}(\Omega) \quad \text{and} \quad \fint_E \alpha_z \cdot \nu_E \, ds = 0 \quad \text{for all } E \notin \mathcal{E}(z).$$

  For any edge E touching z we choose a normal vector $\hat\nu_E$ with counterclockwise orientation (see Figure 5.1 and choose

$$\int_E \alpha_z \cdot \hat\nu_E \, ds = 1.$$

It is not difficult to check that α_z and α_E belong to Z_h and are linear independent. By a dimension argument (see Exercise A.45) it can then be shown that the functions form a basis of Z_h if the domain is simply connected. Details are worked out in Exercise A.49.

§3. Morley element

We consider a variational problem in the space $H^2_0(\Omega)$, the biharmonic problem. Given $f \in L^2(\Omega)$ for simplicity, it seeks a function u such that

$$\Delta^2 u = f \text{ in } \Omega \quad \text{and} \quad u = \partial u/\partial\nu = 0 \text{ on } \partial\Omega.$$

It is easy to calculate via integration by parts that a sufficiently smooth function $u \in H^2_0(\Omega)$ satisfies

$$\int_\Omega \Delta^2 u \varphi \, dx = \int_\Omega D^2 u : D^2 \varphi \, dx = \int_\Omega \Delta u \Delta \varphi \, dx \quad \text{for } \varphi \in C^\infty_c(\Omega).$$

The corresponding variational equality

$$\int_\Omega D^2 u : D^2 v \, dx = \int_\Omega f v \, dx \quad \text{for all } v \in H^2_0(\Omega)$$

has a unique solution by the Riesz representation theorem in $H^2_0(\Omega)$. If V_h is a subspace of $H^2_0(\Omega)$, the Galerkin projection is easily defined and standard theory can be used to establish an a priori

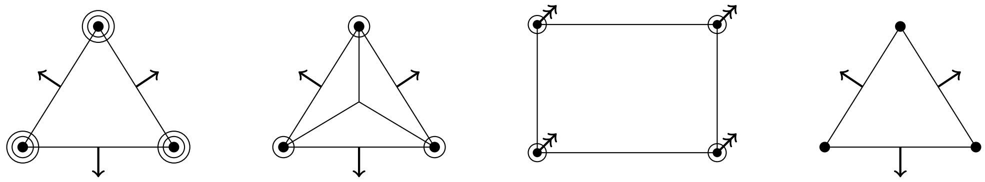


Figure 5.2.: Mnemonic diagrams of some finite elements for the biharmonic equation: Argyris, HCT, BFS (definitions see [Cia78]), and Morley.

error analysis. However, it turns out that the construction of H^2 conforming piecewise polynomial finite element spaces is rather complicated. The three simplest choices are the Argyris element, the Hsieh–Clough–Tocher (HCT) element, or the Bogner–Fox–Schmid (BFS) element from Figure 5.2.

We will use a nonconforming element that allows a much simpler local construction by giving up certain continuity constraints. The Morley element is the following (formal) finite element for a triangle T

$$(T, P_2(T), \{\delta_z, \fint_E \frac{\partial \bullet}{\partial \nu_T}\, ds : z \in \mathcal{N}(T), E \in \mathcal{E}(T)\}),$$

that is, the shape function are the quadratic polynomials and the degrees of freedom are the point evaluations at the three vertices and the evaluations of the averages of the normal derivative over the three edges of the triangle. The Morley finite element space is

$$M_0(\mathcal{T}) := \left\{ v \in P_2(\mathcal{T}) : \begin{array}{l} v \text{ continuous at the interior vertices, } v = 0 \text{ at boundary vertices} \\ \partial v/\partial \nu_E \text{ continuous at the interior edges' midpoints,} \\ \partial v/\partial \nu = 0 \text{ at boundary edges' midpoints} \end{array} \right\}.$$

Given $f \in L^2(\Omega)$, the discrete problem seeks $u_h \in M_0(\mathcal{T})$ such that

$$\int_\Omega D_h^2 u_h : D_h^2 v_h \, dx = \int_\Omega f v_h \, dx \quad \text{for all } v_h \in M_0(\mathcal{T}).$$

It is easy to check that the left hand side defines a positive definite bilinear form: if the piecewise Hessian $D_h^2 v_h$ of v_h is zero, then v_h must be piecewise affine. The continuity at interior vertices implies then that v_h is continuous and thus in $S^1(\mathcal{T})$. The continuity of the normal derivatives over the edge midpoints shows that v_h must be globally affine and, by the boundary conditions imposed on $M_0(\mathcal{T})$, therefore is the zero function. Hence, there exists a unique solution u_h to the discrete problem. The main tool in the error analysis is again a nonconforming interpolation operator, which is defined via the degrees of freedom. Given $v \in H_0^2(\Omega)$, the element $I_h^M v \in M_0(\mathcal{T})$ is uniquely defined by the conditions

$$(v - I_h^M v)(z) = 0 \text{ for all } z \in \mathcal{N} \quad \text{and} \quad \fint_E \frac{\partial(v - I_h v)}{\partial \nu_E}(z) = 0 \text{ for all } E \in \mathcal{E}.$$

With arguments similar to those for the Crouzeix–Raviart element we show the projection property for the Hessian

$$D_h^2 I_h^M v = \Pi_0 D^2 v.$$

There is indeed a close connection between the Morley and the Crouzeix–Raviart method. First, it is directly verified that

$$\nabla_h M_0(\mathcal{T}) \subseteq CR_0^1(\mathcal{T}) \quad \text{and} \quad I_h^{CR} \nabla v = \nabla_h I_h^M v.$$

$$\begin{tikzcd}
H_0^2(\Omega) \arrow[r, "\mathrm{Curl}"] \arrow[d, "I_h^M"'] & {[H_0^1(\Omega)]^2} \arrow[r, "\mathrm{div}"] \arrow[d, "I_h^{CR}"'] & L_0^2(\Omega) \arrow[d, "\Pi_0"'] \\
M_0(\mathcal{T}) \arrow[r, "\mathrm{Curl}_h"] & {[CR_0^1(\mathcal{T})]^2} \arrow[r, "\mathrm{div}_h"] & P_0(\mathcal{T}) \cap L_0^2(\Omega)
\end{tikzcd}$$

Figure 5.3.: Curl-div complex.

We will now prove that the horizontal sequences in Figure 5.3 are exact and that the diagram commutes. We work with the operators

$$\operatorname{Curl} v = \begin{pmatrix} -\partial_y u \\ \partial_x u \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla u \quad \text{and} \quad \operatorname{rot} \phi = \partial_x \phi_2 - \partial_y \phi 1$$

for scalar functions v and vector fields ϕ. The piecewise counterparts are as usual denoted with the index h.

Theorem 5.10. *The diagram of Figure 5.3 commutes. If Ω is simply connected, the horizontal sequences in Figure 5.3 are exact and the diagram commutes.*

Proof. The commuting property is a direct consequence of the projection properties of the respective interpolation operators. It is a classical result that the first row is an exact sequence and we are left with showing this property for the second row. Clearly, $\operatorname{div}_h \operatorname{Curl}_h = 0$, which implies the complex property

$$\operatorname{Curl}_h M_0(\mathcal{T}) \subseteq Z_h$$

where

$$Z_h := \{v_h \in CR_0^1(\mathcal{T}) : \operatorname{div}_h v_h = 0\}.$$

For showing $\operatorname{Curl}_h M_0(\mathcal{T}) = Z_h$ it suffices to compare dimensions. We have previously shown that the dimension of Z_h equals $\operatorname{card}(N(\Omega)) + \operatorname{card}(\mathcal{E}(\Omega))$. This is precisely the number of degrees of freedom of the Morley element and thus the dimension of $M_0(\mathcal{T})$. The kernel of Curl_h, namely the piecewise constant functions, has only a trivial intersection with $M_0(\mathcal{T})$. ∎

From the above we observe that the solution u to the Stokes system with right-hand side f can be written as $\operatorname{Curl} \phi$ for some $\phi \in H_0^2(\Omega)$. We then have $-\Delta \operatorname{Curl} \phi + \nabla p = f$. Taking rot of the equation leads to

$$\Delta^2 \varphi = -\operatorname{rot} f$$

because $\operatorname{rot} \Delta \operatorname{Curl} = \Delta^2$. If the distribution $\operatorname{rot} f$ is an L^2 function, this can be directly discretized with the Morley element. Alternatively, we can discretize the right-hand side with the linear form

$$\int_\Omega f \cdot \operatorname{Curl}_h v_h \, dx \quad \text{for } v_h \in M_0(\mathcal{T}).$$

The resulting method with produce $u_h = \operatorname{Curl}_h \varphi_h$, which is the solution to the Crouzeix–Raviart method. In this sense, the structure from the continuous setting is preserved by the nonconforming spaces. In fluid mechanics, the function ϕ is called *stream function*.

§4. The Helmholtz decomposition

The Helmholtz theorem is a classical result stating that any (unstructured) L^2 vector field can be decomposed as a gradient field and a divergence-free field. In what follows, we denote

$$\mathfrak{Z} = H(\operatorname{div}^0, \Omega) = \{\tau \in H(\operatorname{div}, \Omega) : \operatorname{div} \tau = 0\}.$$

Lemma 5.11 (Helmholtz decomposition). *Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded, connected Lipschitz domain and let $p \in [L^2(\Omega)]^n$. Then there exist a unique $\alpha \in H_0^1(\Omega)$ and a unique $R \in \mathfrak{Z}$ such that*

$$p = \nabla\alpha + R.$$

The decomposition is $L^2(\Omega)$-orthogonal.

Proof. Let $\alpha \in H_0^1(\Omega)$ denote the solution to $-\Delta\alpha = -\operatorname{div} p$ and set $R := p - \nabla\alpha$. Then we have $\operatorname{div} R = 0$ and thus the claimed decomposition. The orthogonality is easily checked with integration by parts, $\int_\Omega \nabla\alpha \cdot R\, dx = -\int_\Omega \alpha \operatorname{div} R\, dx = 0$. ■

In shorthand notation, we write

$$[L^2(\Omega)]^n = \nabla H_0^1(\Omega) \oplus \mathfrak{Z}.$$

The gradient part $\nabla\alpha$ is sometimes called *Helmholtz projector* in the literature. A remarkable structure of the nonconforming method is that is satisfies a discrete analogue of the Helmholtz decomposition. Thereby, we also find a close connection to the Raviart–Thomas space. We will prove

$$[P_0(\mathcal{T})]^n = \nabla_h CR_0^1(\Omega) \oplus \mathfrak{Z}_h$$

where $\mathfrak{Z}_h := RT_0(\mathcal{T}) \cap \mathfrak{Z}$.

Lemma 5.12 (discrete Helmholtz decomposition). *Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded, connected Lipschitz polytope and let $p_h \in [P_0(\mathcal{T})]^n$. Then there exist a unique $\alpha_h \in CR_0^1(\Omega)$ and a unique $R_h \in \mathfrak{Z}_h$ such that*

$$p_h = \nabla_h\alpha_h + R_h.$$

The decomposition is $L^2(\Omega)$-orthogonal.

Proof. As in the continuous case, we denote by $\alpha_h \in CR_0^1(\mathcal{T})$ the unique solution to

$$\int_\Omega \nabla_h\alpha_h \cdot \nabla_h v_h\, dx = \int_\Omega p_h \cdot \nabla_h v_h\, dx \quad \text{for all } v_h \in CR_0^1(\mathcal{T})$$

and denote $R_h := p_h - \nabla_h\alpha_h$. Clearly, R_h is piecewise constant. We denote by ψ_F the Crouzeix–Raviart basis function with respect to the interior face $F \in \mathcal{F}(\Omega)$ (in 2d this is an interior edge). With this test function observe from the above solution property and integration by parts that

$$0 = \int_\Omega R_h \cdot \nabla_h\psi_F\, dx = \int_F [R_h]_F \cdot \nu_F \psi_F\, ds = [R_h]_F \cdot \nu_F \int_F \psi_F\, ds.$$

We conclude that R_h does not have normal jumps and therefore belongs to $H(\operatorname{div}, \Omega)$. Hence, $R_h \in \mathcal{Z}_h$. The orthogonality of the decomposition follows from integration by parts:

$$\int_\Omega \nabla_h\alpha_h \cdot R_h\, dx = -\int_\Omega \alpha_h \operatorname{div} R_h\, dx + \sum_{F\in\mathcal{F}(\Omega)} R_h \cdot \nu_F[\alpha_h]_F\, ds = 0$$

because the jumps of α_h have vanishing integral mean over the faces. ■

Instead of working with explicit gradients, we can equivalently work with the orthogonal complement of $\mathfrak{Z}$ for solving the Poisson equation. We denote by $\Gamma := \nabla H^1_0(\Omega)$ the space of gradients and observe

$$\Gamma = \mathfrak{Z}^\perp.$$

Given $f \in L^2(\Omega)$, assume we are given any vector field $\varphi \in [L^2(\Omega)]^n$ with $-\operatorname{div}\varphi = f$. Then, the Poisson equation $-\Delta u = f$ is equivalent to finding $\gamma \in \Gamma$ with

$$\int_\Omega \gamma \cdot \tau \, dx = \int_\Omega \varphi \cdot \tau \, dx \quad \text{for all } \tau \in \Gamma.$$

The constraint $\gamma \in \Gamma$ can be encoded with a multiplier $z \in \mathfrak{Z}$. The mixed problem is then to find $(\gamma, z) \in [L^2(\Omega)]^n \times \mathfrak{Z}$ such that

$$\begin{aligned} \int_\Omega \gamma \cdot \tau \, dx + \int_\Omega z \cdot \tau \, dx &= \int_\Omega \varphi \cdot \tau \, dx \quad \text{for all } \tau \in [L^2(\Omega)]^n \\ \int_\Omega \gamma \cdot y \, dx &= 0 \quad \text{for all } y \in \mathfrak{Z}. \end{aligned}$$

For showing that this is indeed well-posed, we only need to check the inf-sup condition

$$0 < \beta = \inf_{y \in \mathfrak{Z}\setminus\{0\}} \sup_{\tau \in [L^2(\Omega)]^n \setminus\{0\}} \frac{\int_\Omega y \cdot \tau \, dx}{\|y\|_{H(\operatorname{div},\Omega)} \|\tau\|_{L^2(\Omega)}},$$

which is immediately verified (choose $\tau = y$).

On the discrete level, we can analogously write $\Gamma_h = \nabla_h CR^1_0(\mathcal{T})$ and

$$\Gamma_h = \mathfrak{Z}_h^\perp$$

where now the symbol $\perp$ indicates the orthogonal complement within $[P_0(\mathcal{T})]^n$. The discrete formulation of the above version of Poisson's equation is to find $\gamma_h \in \Gamma_h$ with

$$\int_\Omega \gamma_h \cdot \tau_h \, dx = \int_\Omega \varphi \cdot \tau_h \, dx \quad \text{for all } \tau_h \in \Gamma_h.$$

The mixed problem is then to find $(\gamma_h, z_h) \in [P_0(\mathcal{T})]^n \times \mathfrak{Z}_h$ such that

$$\begin{aligned} \int_\Omega \gamma_h \cdot \tau_h \, dx + \int_\Omega z_h \cdot \tau_h \, dx &= \int_\Omega \varphi \cdot \tau_h \, dx \quad \text{for all } \tau_h \in [P_0(\mathcal{T})]^n \\ \int_\Omega \gamma_h \cdot y_h \, dx &= 0 \quad \text{for all } y \in \mathfrak{Z}_h. \end{aligned}$$

We note that this is a conforming method for the mixed problem (but of course $\Gamma_h \not\subseteq \Gamma$). This shows that the Crouzeix–Raviart method can be interpreted as a conforming method. For a particular choice of φ we can indeed recover the usual Crouzeix–Raviart solution such that $\nabla_h u_h = \gamma_h$.

Lemma 5.13. *Let $f \in L^2(\Omega)$ be piecewise constant. If $\varphi \in RT_0(\mathcal{T})$ with $-\operatorname{div}\varphi = f$ is given as right-hand side in the above mixed problem, then $\nabla_h u_h = \gamma_h$.*

Proof. We can decompose any discrete test function $\tau_h = \nabla_h \alpha_h + R_h$. We conclude from the orthogonality and the solution property of u_h that

$$\int_\Omega \nabla_h u_h \cdot \tau_h \, dx = \int_\Omega \nabla_h u_h \cdot \nabla_h \alpha_h \, dx = \int_\Omega f \alpha_h \, dx.$$

Since $f = -\operatorname{div}\varphi$ and φ is a Raviart–Thomas function, we can integrate by parts

$$\int_\Omega f\alpha_h\,dx = \int_\Omega \varphi\cdot\nabla_h\alpha_h\,dx = \int_\Omega \varphi\cdot\tau_h\,dx - \int_\Omega z_h\cdot\tau_h\,dx$$

where z_h is the orthogonal projection of φ onto $\mathfrak{Z}_h$. Therefore, $\nabla_h u_h$ solves the mixed problem with the multiplier z_h. ∎

Corollary 5.14. *Let f be piecewise constant. Let $\sigma_h \in RT_0(\mathcal{T})$ be the vector part of the mixed Raviart–Thomas solution and let u_h denote the Crouzeix–Raviart solution. Then*

$$\Pi_0\sigma_h = \nabla_h u_h.$$

Proof. It is easy to check that the L^2 projection of σ_h on $\mathfrak{Z}_h$ equals zero (first line of the mixed system) and that $-\operatorname{div}\sigma_h = f$ (second line of the mixed system). In the foregoing proof we have shown

$$\int_\Omega \nabla_h u_h\cdot\tau_h\,dx = \int_\Omega \sigma_h\cdot\tau_h\,dx \quad \text{for all } \tau_h\in[P_0(\mathcal{T})]^n,$$

which is equivalent to the asserted identity. ∎

A. Problems

Exercise A.1. Let L be a linear and continuous map between Banach spaces X, Y. Prove $\ker(L^*) = L(X)^\circ$ and $\ker(L) = {}^\circ(L^*(Y^*))$.

Exercise A.2. Let X, Y be Banach spaces and $L \in \mathcal{L}(X, Y)$. Prove that L is compact if and only if L^* is compact.
Hints (converse direction is similar):
1. An operator is called compact if it maps bounded sets to relatively compact sets.
2. Show that $A = \overline{L(B_1(0))}$ is compact if L is compact.
3. Given a bounded sequence in Y^*, show that it is uniformly bounded and equicontinuous over A. Show that there is a convergent subsequence in $C(A)$ (Arzelà–Ascoli).
4. Show that L^* maps that subsequence to a Cauchy sequence in X^*.

Exercise A.3. For Hilbert spaces X, Y and a continuous linear map $L \in \mathcal{L}(X, Y)$, the map $L^H \in L(Y, X)$ defined by

$$\langle Lx, y\rangle_Y = \langle x, L^H y\rangle_X \quad \text{for any } x \in X, y \in Y$$

is called the *adjoint* of L. Prove

$$L^H = J_X^{-1} \circ L^* \circ J_Y$$

where J_X, J_Y denote the canonical isomorphisms to the biduals of X, Y.
(Hint: Riesz representation theorem.)

Exercise A.4 (Lax–Milgram lemma). Let X be a real Hilbert space with inner product $\langle \cdot, \cdot\rangle_X$ and let $a : X \times X \to \mathbb{R}$ be a bilinear form satisfying the following two properties

- $\exists \beta > 0\, \forall (x, y) \in X^2 \quad |a(x, y)| \le \beta \|x\|_X \|y\|_Y \quad$ (continuity)
- $\exists \alpha > 0\, \forall x \in X \quad \alpha \|x\|_X^2 \le a(x, x) \quad$ (coercivity) .

Prove (using the Banach–Babuška–Nečas lemma) that there exists a unique map $T : X \to X$ with the property

$$a(x, y) = \langle Tx, y\rangle_X \quad \text{for all } (x, y) \in X^2.$$

The map T is linear, continuous, and invertible with

$$\|T\|_{L(X,X)} \le \beta \quad \text{and} \quad \|T^{-1}\|_{L(X,X)} \le \frac{1}{\alpha}.$$

Exercise A.5 (computing with dual spaces)**.** (a) Let $M \subseteq X$ be a subset of a Banach space X. Prove that M° is closed. *(Hint: Embedding in the bidual space.)*
(b) Let $L \in \mathcal{L}(X, Y)$ be a linear and continuous map between Banach spaces X, Y. Prove $\overline{L^*(Y^*)} = {}^\circ(\ker(L^{**})^\circ)$.
(c) Let $L \in \mathcal{L}(X, Y)$ be injective with $L(X) \subseteq Y$ dense. Prove that $L^*(Y^*) \subseteq X^*$ is dense.

Exercise A.6. Prove that any closed subspace of a reflexive Banach space is reflexive.

Exercise A.7 (computing with the orthogonal complement)**.** Let X be a Hilbert space with the Riesz isomorphism $T : X \to X^*$.

(a) Prove that $Z^\perp = T^{-1}(Z^\circ)$ for any closed subspace $Z \subseteq X$.

(b) Let $B : X \to M^*$ be a linear map such that B^* has a bounded inverse on its range. Prove that $B : (\ker B)^\perp \to M^*$ is an isomorphism.

Exercise A.8. Let $g \in H^{1/2}(\partial\Omega)$. Prove that the minimal extension, that is $u \in H^1(\Omega)$ with

$$\|\nabla u\|_{L^2(\Omega)} = \min_{\substack{v \in H^1(\Omega) \\ v|_{\partial\Omega} = g}} \|\nabla v\|_{L^2(\Omega)},$$

is given by the solution to

$$-\Delta u = 0 \quad \text{in } \Omega \quad \text{and} \quad u = g \quad \text{on } \partial\Omega.$$

Exercise A.9 (negative Sobolev space)**.** We know that $H^1_0(\Omega)$ is a Hilbert space when equipped with the inner product $\int_\Omega \nabla v \cdot \nabla w \, dx$. As such, it can be identified with its dual $H^{-1}(\Omega)$. We also know that $L^2(\Omega) \subseteq H^{-1}(\Omega)$. Does this imply that $L^2(\Omega)$ is also a subset of $H^1_0(\Omega)$? Give a complete explanation of this matter.

Exercise A.10 (Gelfand triplet)**.** Following the chain of the Gelfand triplet, we observe that, comparing with Y, a "smaller" space $X \subseteq Y$ will yield a "larger" dual space $Y^* \subseteq X^*$. If X is finite-dimensional $\dim(X) = n$, we know that also $\dim(X^*) = n$. Is therefore Y^* necessarily finite-dimensional?

Exercise A.11. Let $\mathcal{T}$ be a regular triangulation of $\Omega \subseteq \mathbb{R}^n$ and let $v \in [P_1(\mathcal{T})]^n$ be a piecewise affine vector field. For each interior edge F with adjacent triangles T_+ and T_- (i.e., $F = T_+ \cap T_-$), the jump across F is defined by $[v]_F := v|_{T_+} - v|_{T_-}$. Prove that

$$v \in H(\mathrm{div}, \Omega) \iff [v \cdot \nu_F]_F = 0 \quad \text{for all interior edges } F$$

where ν_F is some normal vector of F.

Exercise A.12. Prove that the normal trace is a surjective map from $\{v \in H(\mathrm{div}, \Omega) : \mathrm{div}\, v = 0\}$ to $\{g \in H^{-1/2}(\partial\Omega) : \langle g, 1 \rangle = 0\}$.

Exercise A.13. Prove that the mixed form of the Poisson equation satisfies the properties of the Brezzi splitting theorem.

Exercise A.14. Prove that the local basis functions $\psi_{T,E}$ satisfy the property (2.1).

Exercise A.15. Write a routine (Python or pseudocode) that provides a global enumeration of all edges in a given mesh $\mathcal{T}$.

Exercise A.16. Implement the mixed Raviart–Thomas method for the homogeneous Dirichlet problem of the Laplacian. Use the data from earlier exercises to compute experimental rates of convergence in different norms.

Exercise A.17. Let T be a triangle. Prove that the following triplets $(T, \mathcal{P}, \mathcal{L})$ are finite elements in the sense of Ciarlet.

- The cubic Lagrange element: $\mathcal{P} = P_3(T)$ and $\mathcal{L}$ contains the point evaluations in the three vertices of T, in two interior points of each edge, and in the midpoint of T.
- The Crouzeix–Raviart element: $\mathcal{P} = P_1(T)$ and $\mathcal{L} := \{\fint_E \cdot\, dx : E \in \mathcal{E}(T)\}$.
- The cubic Hermite element: $\mathcal{P} = P_3(T)$ and $\mathcal{L}$ contains the point evaluations in the three vertices and in the midpoint of T and the evaluation of the gradient in the vertices, that is

$$\mathcal{L} = \{v \mapsto v(z) : z \in \mathcal{N}(T)\} \cup \{v \mapsto \nabla v(z) : z \in \mathcal{N}(T)\} \cup \{v \mapsto v(\mathrm{mid}(T))\}.$$

- The Argyris element: $\mathcal{P} = P_5(T)$ and

$$\mathcal{L} = \{v \mapsto v(z), v \mapsto \nabla v(z), v \mapsto D^2 v(z) : z \in \mathcal{N}(T)\} \cup \{v \mapsto \fint_E \nabla v \cdot \nu_T \, ds : E \in \mathcal{E}(T)\}.$$

Exercise A.18. Let T be a triangle and let $\mathcal{L}$ consist of the six functionals

$$\{\fint_E \cdot\, ds : E \in \mathcal{E}(T)\} \cup \{\fint_E \cdot s \, ds : E \in \mathcal{E}(T)\}$$

describing the first-order moments of a function over the three edges. Prove that $(T, P_2(T), \mathcal{L})$ is not a finite element in the sense of Ciarlet.

Exercise A.19. Let $\mathcal{P}$ be an m-dimensional vector space and let $\mathcal{F}$ be a subset of $\mathcal{P}^*$ with m elements. Prove that the elements of $\mathcal{F}$ form a basis of $\mathcal{P}^*$ if and only if for any $v \in \mathcal{P}$ the relation $\langle F, v \rangle = 0$ for all $F \in \mathcal{F}$ implies $v = 0$.

Exercise A.20. Prove that there exists a constant that only depends on the shape regularity such that

$$\|v - I_{RT} v\|_{L^2(T)} \leq C h_T \|Dv\|_{L^2(T)} \quad \text{for any } v \in [H^1(T)]^2$$

and

$$\|\operatorname{div}(v - I_{RT} v)\|_{L^2(T)} \leq C h_T \|\nabla \operatorname{div} v\|_{L^2(T)} \quad \text{for any } v \in [H^2(T)]^2.$$

Exercise A.21. Prove that unit normal vectors transform as

$$\nu(x) = \frac{1}{|B^{-\top} \hat{\nu}(\hat{x})|} B^{-\top} \hat{\nu}(\hat{x}).$$

Exercise A.22. Prove Lemma 2.17.

Exercise A.23. Prove that the Raviart–Thomas interpolation is invariant under the Piola transform, i.e.,

$$I_{RT,\hat{T}} \hat{q} = \widehat{I_{RT,T} q}.$$

Exercise A.24. Let $u(x) = \operatorname{sign}(x_1)$ and let $\Omega = (-1,1)^n$ denote the hypercube in $\mathbb{R}^n$. Prove that $u \in H^s(\Omega)$ if $0 < s < 1/2$ and that $u \notin H^s(\Omega)$ if $1/2 \leq s < 1$.

Exercise A.25. Let $0 < s < 1$ and let $u \in H^s(\Omega)$ have compact support in Ω. Denote $\delta = \operatorname{dist}(\operatorname{supp}(u), \partial\Omega)$ and let $\tilde{u}$ denote the continuation of u by zero to $\mathbb{R}^n$. Prove that $\tilde{u} \in H^s(\mathbb{R}^n)$ and

$$\|\tilde{u}\|^2_{H^s(\mathbb{R}^n)} \leq C(1 + s^{-1} \delta^{-2s}) \|u\|^2_{H^s(\Omega)}.$$

Hint: Lemma 6.34 in [Dob10].

Exercise A.26. Let $u : \mathbb{R}^n \to \mathbb{R}$ be of class C^1 and let $x, y \in \mathbb{R}^n$. Prove that

$$u(z) - u(x) = \int_0^1 \nabla(tz + (1-t)x) \cdot (z - x)\, dt.$$

Exercise A.27. Convince yourself that locally flattening the boundary of a Lipschitz domain preserves the H^1 property. Consult Lemma 6.6 from [Dob10]

Exercise A.28. Let $u \in \widetilde{H}^{1/2}(\Omega)$. Prove that

$$\|u\|^2_{\widetilde{H}^{1/2}(\Omega)} = \|u\|^2_{H^{1/2}(\Omega)} + 2\int_\Omega |u(x)|^2 \int_{\mathbb{R}^n \setminus \Omega} |x - y|^{-n-1}\, dy\, dx.$$

Exercise A.29. Prove that the bilinear form $(v, w) \mapsto \int_0^1 v'(x)w(x)\, dx$ does not possess a continuous extension to $H^{1/2}((0,1)) \times H^{1/2}((0,1))$. *Hint: Consider the function* $\log(|\log(x/\exp(1))|)$.

Exercise A.30. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open Lipschitz polygon. Prove that the tangential derivative ∂_s is a continuous map from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$.

Exercise A.31. On the L-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1] \times [-1,0])$ we are given the Dirichlet boundary $\Gamma_D = \{0\} \times [-1,0] \cup [0,1] \times \{0\}$ and the Neumann boundary $\Gamma_N = \partial\Omega \setminus \Gamma_D$. For boundary data $u_D = 1$ on Γ_D, and Neumann data

$$g(x,y) = \frac{2}{3} r^{-1/3} \times \begin{cases} \cos(\phi)\sin(2\phi/3) - \sin(\phi)\cos(2\phi/3) & \text{if } x = 1 \\ \sin(\phi)\sin(2\phi/3) + \cos(\phi)\cos(2\phi/3) & \text{if } y = 1 \\ -\cos(\phi)\sin(2\phi/3) + \sin(\phi)\cos(2\phi/3) & \text{if } x = -1 \\ -\sin(\phi)\sin(2\phi/3) - \cos(\phi)\cos(2\phi/3) & \text{if } y = -1 \end{cases}$$

in polar coordinates (r, ϕ), and $f = 0$, solve the mixed boundary value problem for the Laplacian with the mixed Raviart–Thomas FEM. The exact solution is given by $u(r,\phi) = 1 + r^{2/3}\sin(2\phi/3)$. Plot the convergence history for the L^2 norm of $u - u_h$ as well as $\sigma - \sigma_h$ and on $\Pi_h u - u_h$.

Exercise A.32. Let $\Omega \subset \mathbb{R}^2$ be a connected and open polygonal domain. Prove that every $u \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfies the identity

$$\|\Delta u\|^2 = \|D^2 u\|^2.$$

Conclude that $\Delta : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ is injective with closed range. Furthermore, prove that exists a constant $C(\Omega)$ such that every $u \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfies

$$\|u\|_{H^2(\Omega)} \leq C(\Omega)\|\Delta u\|.$$

Exercise A.33. Consider the ODE

$$v''(r) + r^{-1}v'(r) - \lambda^2 r^{-2} v(r) = 0 \qquad 0 < r < \rho$$

for some nonnegative real number λ. Prove that the solution is given by

$$v(r) = \begin{cases} \alpha r^\lambda + \beta r^{-\lambda} & \text{if } \lambda > 0 \\ \alpha + \beta\log(r) & \text{if } \lambda = 0 \end{cases}$$

with real numbers α, β. *Hint: The ODE is called Cauchy–Euler equation.*

Exercise A.34. We consider $v \in H(\Delta, \Omega)$ for a polygon Ω. Let $\Gamma \subseteq \partial\Omega$ be a straight segment of the boundary. Prove that $v|_\Gamma \in [\widetilde{H}^{1/2}(\Gamma)]^*$. *Hint: You may use [Gri85] that the trace of* $\partial \cdot /\partial\nu$ *is continuous and onto from* $H_0^1(\Omega) \cap H^2(\Omega)$ *to* $\widetilde{H}^{1/2}(\Omega')$ *for any convex polygon* Ω'.

Exercise A.35. Let $\Omega \subset \mathbb{R}^2$ be a connected and open polygonal domain. Prove that $H^2(\Omega)$ is dense in $H(\Delta, \Omega)$, but $H_0^1(\Omega) \cap H^2(\Omega)$ is not dense in $H_0^1(\Omega) \cap H(\Delta, \Omega)$.

Exercise A.36. Show that in two dimensions and for $0 < s, \alpha < 1$, we have $r^\alpha \in H^{1+s}(\Omega)$ if and only if $s < 1 + \alpha$. Prove Theorem 4.10.

Exercise A.37. Prove Lemma 4.8.

Exercise A.38. Let $u_h \in S_0^1(\mathcal{T})$ be the standard FEM solution to the right-hand side $f \in L^2(\Omega)$. Let z be an interior vertex of $\mathcal{T}$ with hat function φ_z. Prove that

$$\frac{1}{2} \sum_{E \in \mathcal{E}(z)} \int_E [\nabla u_h]_E \cdot \nu_E \, ds = \int_{\omega_z} f \varphi_z \, dx$$

for the set $\mathcal{E}(z)$ of edges containing z.

Exercise A.39. Prove that for $f \in L^2(\Omega)$ the following error bound holds

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq \|\tau_h^{\mathrm{pw}}\|_{L^2(\Omega)} + \sqrt{\sum_{T \in \mathcal{T}} \frac{h_T^2}{\pi^2} \|f - \fint_T f \, dx\|_{L^2(T)}^2}.$$

Hint: You may use that the Poincaré constant on a convex domain ω *can be bounded by* $\mathrm{diam}(\omega)/\pi$.

Exercise A.40. Consider the Dirichlet problem for the Laplacian with homogeneous boundary conditions on the unit square. with $f(x) = 2(x_1(1 - x_1) + x_2(1 - x_2))$ and exact solution $u(x) = x_1(x_1 - 1)x_2(x_2 - 1)$. Compute $\|\nabla u_h - \sigma_h\|_{L^2(\Omega)}$ on a sequence of mesh refinements and compare with the true errors.

Exercise A.41. Implement the error estimator $\|\tau_h^{\mathrm{pw}}\|_{L^2(\Omega)}$ and test its performance for the setting of the previous Exercise.

Exercise A.42. Prove the projection property $\nabla_h I_h = \Pi_0 \nabla$ for the nonconforming interpolation operator.

Exercise A.43. Prove that $|\!|\!|\cdot|\!|\!|_h$ is a norm on $H_0^1(\Omega) + CR_0^1(\mathcal{T})$.

Exercise A.44. Let T be a triangle and $v \in H^1(T)$ satisfy $\fint_E v \, ds = 0$ for an edge E of T. Prove

$$\|v\|_{L^2(T)} + h_T^{1/2} \|v\|_{L^2(E)} \leq C h_T \|\nabla v\|_{L^2(T)}$$

with a constant C that only depends on the shape regularity.

Exercise A.45 (Euler formulae)**.** Let $\mathcal{T}$ be a triangulation of the simply-connected polygonal domain Ω. Prove

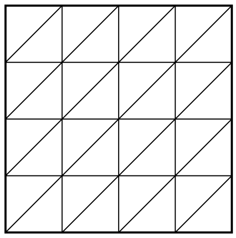
$$\mathrm{card}(\mathcal{T}) + \mathrm{card}(\mathcal{N}) = 1 + \mathrm{card}(\mathcal{E}) \quad \text{and} \quad 2\,\mathrm{card}(\mathcal{T}) + 1 = \mathrm{card}(\mathcal{N}) + \mathrm{card}(\mathcal{E}(\Omega)).$$

Here, as usual, $\mathcal{E}$ is the set of edges, $\mathcal{E}(\Omega)$ the set of interior edges, and $\mathcal{N}$ the set of vertices. What happens on planar domains with holes?

Exercise A.46. Implement the Crouzeix–Raviart method for Ω and f as in Exercise A.40 and produce experimental convergence history plots for the error in the $|||\cdot|||_h$ norm and the L^2 norm.

Exercise A.47. Prove that the L^2 error of the Crouzeix–Raviart method is of order h^2 provided $u \in H_0^1(\Omega) \cap H^2(\Omega)$.

Exercise A.48. *(conforming divergence-free functions are trivial)*
Let $\mathcal{T}$ be the criss triangulation of the unit square and let $u_h \in [S_0^1(\mathcal{T})]^2$ with $\operatorname{div} u_h = 0$. Prove that $u_h = 0$.
Hint: The criss triangulation is



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Exercise A.49. Prove that the functions α_z, α_E for all interior vertices z and interior edges E form a basis of Z_h if Ω is simply connected. How needs the construction be modified for domains with holes?

Exercise A.50. Implement the Crouzeix–Raviart method for the Stokes equations. As a test example, use the following data on the square $\Omega = (-1, 1)^2$ (not the unit square): The right-hand side $f = 0$ is zero and the exact solution is

$$u(x_1, x_2) = \begin{pmatrix} 20x_1x_2^4 - 4x_1^5 \\ 20x_1^4x_2 - 4x_2^5. \end{pmatrix}$$

Choose the inhomogeneous Dirichlet data u_D according to u. Create convergence history plots for the $|||\cdot|||_h$ error in the u variable and the L^2 error in the p variable.

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