Viscosity solutions — Problem sheet 1

Problem 1. Let a linear 2nd order PDE be given by

$$\sum_{|\alpha|\leq 2} a_\alpha(x)\partial^\alpha u(x) = f(x).$$

Here, a_α and f are given functions over Ω. The above sum runs over all multi-indices α of length ≤ 2, and ∂^α is the partial derivative with respect to α. Show that the linear PDE is degenerate ellitpic if and only if the matrix $(a_\alpha)_{|\alpha|=2}$ of the indices belonging to multiindices of length 2 is positive semi-definite.

Problem 2. Write the following PDEs in the format $F(x, u(x), \nabla u(x), D^2u(x)) = 0$ and decide which of them are linear or degenerate elliptic.

- Poisson's equation
- ∞-Poisson equation
- heat equation
- $\partial^2 u_{tt} - \partial^2 u_{xx} = f(t, x)$ for $(t, x) \in \mathbb{R} \times \mathbb{R}$
- $\partial_{x_1} u(x) - u(x)\,\mathrm{tr}(\left(\begin{smallmatrix} 2 & 1 \\ 0 & -10 \end{smallmatrix}\right) D^2u(x)) = f(x)$ for $x = (x_1, x_2) \in \mathbb{R}^2$
- $\Delta u(x) - |u(x)|^3 = 0$ for $x \in \mathbb{R}^3$

Problem 3. Let $u \in C^2(\mathbb{R}^n)$. Prove that

$$\mathcal{J}^{2,+}u(x) \cap \mathcal{J}^{2,-}u(x) = \{(\nabla u(x), D^2u(x))\}$$

for every $x \in \mathbb{R}^n$.

Problem 4. Let $W : \mathbb{R} \to [0, 1]$ be a bounded function that is nowhere differentiable and define the function $w \in C(\mathbb{R})$ by $w(x) = W(x)|x|^3$.

(a) Prove that w is differentiable in $x = 0$.

(b) Prove that w is nowhere differentiable in $\mathbb{R}\backslash\{0\}$ and that w is not twice differentiable in $x = 0$.

(c) Prove that $w(z) = o(|z|^2)$ as $z \to 0$.

(d) Prove that $(0_{\mathbb{R}^n}, 0_{\mathbb{S}^{n\times n}}) = \mathcal{J}^{2,+}w(0) \cap \mathcal{J}^{2,-}w(0)$.

Problem 5. Let $u(x) = -|x|$. Compute $\mathcal{J}^{2,\pm}u(x)$ for every $x \in \mathbb{R}$. Draw a picture of representative elements of the upper and lower semijets of in the point $x = 0$.

Problem 6. Check the following real functions for upper/lower semicontinuity in $x = 0$.

$$u(x) = \begin{cases} x/|x| & x \neq 0 \\ 1 & \text{else} \end{cases}, \quad v(x) = \begin{cases} x^{-1} & x > 0 \\ 0 & \text{else} \end{cases}, \quad w(x) = \begin{cases} \cos(x^{-1}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$