Viscosity solutions — Problem sheet 2

Problem 7. Let $K \subseteq \mathbb{R}^n$ be compact. Prove that any $v \in USC(K)$ attains its maximum and any $w \in LSC(K)$ attains its minimum.

Problem 8. Given a function $u : \Omega \to \mathbb{R}$, prove that $u^* \in USC(\Omega)$.

Problem 9. Let $u : \Omega \to \mathbb{R}$. Prove that $u \leq u^*$. Prove further that any $v \in USC(\Omega)$ with $v \geq u$ satisfies $v \geq u^*$. *(The comparison of functions is meant in the pointwise sense.)*

Problem 10. Compute the envelopes w^* and w_* of

$$w(x) = \begin{cases} \cos(x^{-1}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Problem 11 (lack of comparison)**.** Consider the one-dimensional elliptic Dirichlet problem

$$u''(x) + 18x(u'(x))^4 = 0 \quad \text{in } (-1,1), \quad u(-1) = b, \quad u(1) = -b$$

for some $b > 1$. Let furthermore the functions

$$\underline{u}(x) = \begin{cases} x^{1/3} - 1 + b, & x \in [0,1], \\ x^{1/3} + 1 - b, & x \in [-1,0), \end{cases} \qquad \bar{u}(x) = \begin{cases} x^{1/3} - 1 + b, & x \in (0,1], \\ x^{1/3} + 1 - b, & x \in [-1,0], \end{cases}$$

be given. Prove that $\underline{u}$ is subsolution and $\bar{u}$ is supersolution to the Dirichlet problem but $\underline{u}(z) > \bar{u}(z)$ for some point z.

Problem 12. Let $\Omega \subseteq \mathbb{R}^n$ be open, $x \in \bar{\Omega}$, $u \in USC(\bar{\Omega})$, and $\psi \in C^2(\bar{\Omega})$. Prove that

$$\mathcal{J}^{2,+}(u+\psi)(x) = \left\{ (\nabla\psi(x), D^2\psi(x)) + (p,X) : (p,X) \in \mathcal{J}^{2,+}u(x) \right\}$$

and

$$\bar{\mathcal{J}}^{2,+}(u+\psi)(x) = \left\{ (\nabla\psi(x), D^2\psi(x)) + (p,X) : (p,X) \in \bar{\mathcal{J}}^{2,+}u(x) \right\}.$$

Problem 13. Let $u \in USC(\Omega)$ have a local maximum at $x \in \Omega$. Prove that any symmetric positive semidefinite matrix $0 \leq A \in \mathbb{S}^{n\times n}$ satisfies

$$(0, A) \in \mathcal{J}^{2,+}u(x).$$

Problem 14. Let $u, v \in USC(\Omega)$ and $\psi \in C^2(\Omega \times \Omega)$. Assume the function $u(x) + v(x) - \psi(x,y)$ has a maximum at $(x,y) \in \Omega \times \Omega$. Prove that there exist $p_1, p_2 \in \mathbb{R}^n$ and $B \in \mathbb{S}^{2n\times 2n}$ such that

$$((p_1,p_2), B) \in \mathcal{J}^{2,+}(u(x)+v(y)) \quad \text{and} \quad B \leq D^2\psi(x,y).$$