Viscosity solutions — Problem sheet 3

Problem 15. Prove that uniform ellipticity implies degenerate ellipticity.

Problem 16 (properties of Pucci's operators I)**.** Let $0 < \lambda \leq \Lambda$ and $M \in \mathbb{S}^{n\times n}$. Let the eigenvalues of M be denoted by $\alpha_1, \dots, \alpha_n$. Prove that

$$\mathcal{P}^-(M,\lambda,\Lambda) = \lambda \sum_{\alpha_j>0} \alpha_j + \Lambda \sum_{\alpha_j<0} \alpha_j \quad \text{and} \quad \mathcal{P}^+(M,\lambda,\Lambda) = \Lambda \sum_{\alpha_j>0} \alpha_j + \lambda \sum_{\alpha_j<0} \alpha_j.$$

Problem 17 (properties of Pucci's operators II)**.** Let $M, N \in \mathbb{S}^{n\times n}$. Prove

1. $\mathcal{P}^-(M) \leq \mathcal{P}^+(M)$
2. $\mathcal{P}^-(M,\lambda',\Lambda') \leq \mathcal{P}^-(M,\lambda,\Lambda)$ and $\mathcal{P}^+(M,\lambda',\Lambda') \geq \mathcal{P}^-(M,\lambda',\Lambda')$ if $\lambda' \leq \lambda \leq \Lambda \leq \Lambda'$
3. $\mathcal{P}^-(M) = -\mathcal{P}^+(-M)$
4. $\mathcal{P}^\pm(\alpha M) = \alpha \mathcal{P}^\pm(M)$ if $\alpha \geq 0$
5. $\mathcal{P}^+(M) + \mathcal{P}^-(N) \leq \mathcal{P}^+(M+N) \leq \mathcal{P}^+(M) + \mathcal{P}^+(N)$
6. $\mathcal{P}^-(M) + \mathcal{P}^-(N) \leq \mathcal{P}^-(M+N) \leq \mathcal{P}^-(M) + \mathcal{P}^+(N)$
7. $\lambda\|N\| \leq \mathcal{P}^-(N,\lambda,\Lambda) \leq \mathcal{P}^+(N,\lambda,\Lambda) \leq n\Lambda\|N\|$ if $N \geq 0$
8. $\mathcal{P}^-$ and $\mathcal{P}^+$ are uniformly elliptic with ellipticity constants λ, $n\Lambda$.

Problem 18. For matrices $A, B \in \mathbb{R}^{n\times n}$ we define the Frobenius inner product $A : B = \sum_{j,k=1}^n A_{jk}B_{jk}$. Prove that $A : B = \operatorname{tr}(AB^\top)$ and $x^\top Ay = A : x \otimes y$ for $x, y \in \mathbb{R}^n$. *(Recall $x \otimes y = xy^\top$.)*

Problem 19 (Hamilton–Jacobi–Bellman operator)**.** Let $\mathcal{A}$ be an index set and $0 < \lambda \leq \Lambda$ be given. Let, for any $\alpha \in \mathcal{A}$, $A_\alpha : \Omega \to \mathbb{S}^{n\times n}$ be measurable and bounded $0 < \lambda I \leq A_\alpha \leq \Lambda I$ uniformly in Ω; and let $f_\alpha \in L^\infty(\Omega)$. Prove that the operator

$$F(x, D^2u(x)) = \inf_{\alpha\in\mathcal{A}} \big(\operatorname{tr}(A_\alpha D^2u(x)) - f_\alpha(x)\big)$$

is uniformly elliptic.

Problem 20. Prove Ishii's lemma under the assumption $w_1, w_2 \in C^2(\bar{\Omega})$.