Viscosity solutions — Problem sheet 4

Problem 21. Let $\mathcal{A}$ be a family of affine functions over $\mathbb{R}^n$. Prove that $\sup \mathcal{A}$ is a convex.

Problem 22. Let $\Omega \subseteq \mathbb{R}^n$ an open domain and $f : \Omega \to \mathbb{R}$ be convex. Prove that f is locally Lipschitz continuous.
Instruction (if needed): To show Lipschitz continuity near $x_0 \in \Omega$, let $B_{2r}(x_0) \subseteq \Omega$ be an open ball with $x, y \in B_r(x_0)$ and define $z := x + \alpha(x - y)$ with $\alpha = r/(2|x - y|)$. Show $x = (1 + \alpha)^{-1}z + \alpha(1 + \alpha)^{-1}y$ and use this result to first estimate $f(x) - f(y) \leq (\alpha + 1)^{-1}(f(z) - f(y))$ and then establish the Lipschitz bound. Prove the estimate for $|f(x) - f(y)|$ by interchanging the roles of x, y.

Problem 23. Prove the inclusion

$$\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} K_{\delta,\rho}^{1/k} \subseteq K_{\delta,\rho}$$

claimed in the proof of Jensen's lemma.

Problem 24. Let $X \in \mathbb{S}^{n\times n}$, $a, b \in \mathbb{R}^n$, and $\varepsilon > 0$. As usual, $\|X\| = \max|\sigma(X)|$ is the spectral norm of X (natural matrix norm w.r.t. the Euclidean scalar product).

(i) Prove
$$y^\top Xy = z^\top Xz + (y - z)^\top X(y - z) + 2(y - z)^\top Xz.$$

(ii) Prove that any $a, b \in \mathbb{R}$ satisfy $2ab \leq \varepsilon^{-1}a^2 + \varepsilon b^2$. *(Hint: binomial formula.)*

(iii) Prove
$$y^\top Xy \leq z^\top Xz + \|X\|\, |z - y|^2 + \varepsilon^{-1}|z - y|^2 + \varepsilon|Xz|^2.$$

(iv) Prove
$$y^\top Xy \leq z^\top(X + \varepsilon X^2)z + (\varepsilon^{-1} + \|X\|)|z - y|^2.$$