Research Article

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A Remark on Newest Vertex Bisection in Any Space Dimension

Abstract: With newest vertex bisection, there is no uniform bound on the number of n-simplices that need to be refined to arrive at the smallest conforming refinement $\mathcal{T}'$ of a conforming partition $\mathcal{T}$ in which one simplex has been bisected. In this note, we show that the difference in levels between any $T' \in \mathcal{T}'$ and its ancestor $T \in \mathcal{T}$ *is* uniformly bounded. This result has been used in [2, Lemma 4.2] by Carstensen and the first two authors.

Keywords: Newest-Vertex Bisection, Adaptive Mesh-Refinement, n-Simplices

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1 Newest Vertex Bisection

From [3, 5, 6], we recall the generalization to $n \geq 2$ dimensions of the *newest vertex bisection* algorithm. A *tagged simplex* $(z_0, \dots, z_n; \gamma)$ is an $(n+2)$-tuple of vertices $z_0, \dots, z_n \in \mathbb{R}^n$, which do not lie on an $(n-1)$-dimensional hyperplane, and of a type $\gamma \in \{0, \dots, n-1\}$. The mapping $\operatorname{dom} : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \{0, \dots, n-1\} \to 2^{\mathbb{R}^n}$ extracts the corresponding (closed) simplex $\operatorname{dom}(z_0, \dots, z_n; \gamma) := \operatorname{conv}\{z_0, \dots, z_n\}$ from a tagged simplex $(z_0, \dots, z_n; \gamma)$. For convenience, for a tagged simplex T we often denote $\operatorname{dom}(T)$ simply as T.

The *bisection* of a tagged simplex $(z_0, \dots, z_n; \gamma)$ generates the two tagged simplices

$$\Big(z_0, \frac{z_0 + z_n}{2}, z_1, \dots, z_\gamma, z_{\gamma+1}, \dots, z_{n-1}; (\gamma+1) \bmod n\Big),$$
$$\Big(z_n, \frac{z_0 + z_n}{2}, z_1, \dots, z_\gamma, z_{n-1}, \dots, z_{\gamma+1}; (\gamma+1) \bmod n\Big).$$

(By convention, the finite sequences $(z_{\gamma+1}, \dots, z_{n-1})$ and $(z_1, \dots, z_\gamma)$ are void for $\gamma = n-1$ and $\gamma = 0$, respectively.) The edge $\operatorname{conv}\{z_0, z_n\}$ of the original simplex that has been cut is known as its refinement edge. The two new tagged simplices are called the *children* of the tagged simplex $(z_0, \dots, z_n; \gamma)$, and any child of some child of a tagged simplex is called *grandchild*.

Let $\mathcal{T}_0$ be an *initial*, *conforming* triangulation of a polyhedral bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ into tagged n-simplices. This means that the corresponding set of simplices $\{T : T \in \mathcal{T}_0\}$ covers the domain $\overline{\Omega}$, and that two distinct simplices $T = \operatorname{conv}\{y_0, \dots, y_n\}$ and $T' := \operatorname{conv}\{z_0, \dots, z_n\}$ from $\mathcal{T}_0$ are either disjoint or share exactly one surface (e.g., an edge or side) in the sense that there exist $0 \leq j_1 < \cdots < j_N \leq n$ and $0 \leq k_1 < \cdots < k_N \leq n$ for some $N \in \{1, \dots, n\}$ such that

$$T \cap T' = \operatorname{conv}\{y_{j_1}, \dots, y_{j_N}\} = \operatorname{conv}\{z_{k_1}, \dots, z_{k_N}\}.$$

We will exclusively consider partitions of tagged simplices that are descendants of $\mathcal{T}_0$, meaning that they can be created by recurrent bisections of individual simplices in the triangulation starting from $\mathcal{T}_0$. Such partitions are *uniformly shape regular* in the sense that for any simplex T from any of these partitions

$$\operatorname{meas}(T)^{1/n} \simeq \operatorname{diam}(T) \simeq 2^{-\ell(T)/n}$$

only dependent on $\mathcal{T}_0$. Here $\ell(T)$ denotes the level of T, being the number of bisections that are needed to create T from a simplex T' in $\mathcal{T}_0$. Note that $\ell(T) = \text{meas}(T)/\text{meas}(T')$.

Here and in the following, by $C \lesssim D$ we will mean that C can be bounded by a multiple of D, only dependent on the initial triangulation $\mathcal{T}_0$. Furthermore, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

In view of applications in adaptive finite element methods, more specifically we will restrict our considerations to those triangulations that in addition are *conforming*. The set of all *conforming descendants* of $\mathcal{T}_0$ will be denoted by $\mathbb{T}$.

Using the uniform shape regularity and conformity, one easily shows the following result.

Lemma 1.1. *There exist constants $C, c > 0$ such that*
(a) *for any $T, T' \in \mathcal{T} \in \mathbb{T}$ with $T \cap T' \neq \emptyset$, it holds that $|\ell(T) - \ell(T')| \leq C$;*
(b) *for any $T, T' \in \mathcal{T} \in \mathbb{T}$ with $\ell(T) > \ell(T') + C$, it holds that $\text{dist}(T, T') \geq c2^{-\ell(T')/n}$.*

2 Matching Condition

Note that, given a tagged simplex $T = (z_0, \dots, z_n; \gamma)$, the tagged simplex

$$T_R := (z_n, z_1, \dots, z_\gamma, z_{n-1}, z_{n-2}, \dots, z_{\gamma+1}, z_0; \gamma)$$

with $\text{dom}(T_R) = \text{dom}(T)$ has the same children as T. Two tagged simplices T, T' are called neighbors, if they share a common $(n-1)$-dimensional hyper-surface. Two neighboring tagged simplices T and T' are called *reflected neighbors*, if the ordered sequence of vertices of either T or T_R coincides with that of T' on all but one position; for graphical illustrations cf. [5].

We will impose the following condition on $\mathcal{T}_0$.

Definition 2.1 (Matching condition). All simplices in $\mathcal{T}_0$ are of the same type γ. Any two neighboring tagged simplices $T = (y_0, \dots, y_n; \gamma)$ and $T' = (z_0, \dots, z_n; \gamma)$ in $\mathcal{T}_0$ satisfy the following conditions.
(a) If $\text{conv}\{y_0, y_n\} \subseteq T \cap T'$ or $\text{conv}\{z_0, z_n\} \subseteq T \cap T'$, then T and T' are reflected neighbors.
(b) If $\text{conv}\{y_0, y_n\} \nsubseteq T \cap T' \neq \emptyset$ and $\text{conv}\{z_0, z_n\} \nsubseteq T \cap T'$, then any two neighboring children of T and T' are reflected neighbors.

The matching condition guarantees that all uniform refinements of $\mathcal{T}_0$ are conforming [5, Theorem 4.3], and it is actually needed for this property to hold. For completeness, with a uniform refinement of $\mathcal{T}_0$ we mean a descendant of $\mathcal{T}_0$ in which all simplices have the same level.

3 Refinements

We equip $\mathbb{T}$ with a partial ordering by defining, for $\mathcal{T}, \mathcal{T}' \in \mathbb{T}$, $\mathcal{T} \leq \mathcal{T}'$ when $\mathcal{T}'$ is a refinement of $\mathcal{T}$. With this partial ordering, $(\mathbb{T}, \leq)$ is a *lattice*, i.e., for any $\mathcal{T}, \mathcal{T}' \in \mathbb{T}$, the smallest common refinement $\mathcal{T} \vee \mathcal{T}'$ and greatest common coarsening $\mathcal{T} \wedge \mathcal{T}'$ in $\mathbb{T}$ are well-defined. A characterization of both these partitions is given in the following remark.

Remark 3.1. For $\mathcal{T}, \mathcal{T}' \in \mathbb{T}$, $T \in \mathcal{T}$ and $T' \in \mathcal{T}'$ with $T \subseteq T'$, it holds that $T' \in \mathcal{T} \wedge \mathcal{T}'$ and $T \in \mathcal{T} \vee \mathcal{T}'$, see, e.g., [4, Lemma 4.3].

For $\mathcal{T} \in \mathbb{T}$, and a set $\mathcal{M} \subseteq \mathcal{T}$ (the set of simplices 'marked for refinement'), let

$$\mathcal{T}' := \texttt{refine}(\mathcal{T}, \mathcal{M})$$

denote the *smallest* partition in $\mathbb{T}$ with $\mathcal{T} \leq \mathcal{T}'$ and $\mathcal{M} \cap \mathcal{T}' = \emptyset$. The uniform refinement $\bar{\mathcal{T}}$ of $\mathcal{T}_0$ consisting of all simplices with level equal to $1 + \max_{T \in \mathcal{T}} \ell(T)$ satisfies $\mathcal{T} \leq \bar{\mathcal{T}}$ and $\mathcal{M} \cap \bar{\mathcal{T}} = \emptyset$. Consequently, $\mathcal{T}'$ is well-defined as the greatest common coarsening of the finite, non-empty set $\{\tilde{\mathcal{T}} \in \mathbb{T} : \mathcal{M} \cap \tilde{\mathcal{T}} = \emptyset,\ \mathcal{T} \leq \tilde{\mathcal{T}} \leq \bar{\mathcal{T}}\}$.

The following result was proved in [5, Theorems 5.1–5.2].

Lemma 3.2. *Let $T \in \mathcal{T} \in \mathbb{T}$ and $\mathcal{T}' := \texttt{refine}(\mathcal{T}, \{T\})$. If $T' \in \mathcal{T}'$ is newly created by the call $\texttt{refine}(\mathcal{T}, \{T\})$, i.e., $T' \in \mathcal{T}' \setminus \mathcal{T}$, then*
(a) $\ell(T') \le \ell(T) + 1$,
(b) $\operatorname{dist}(T', T) \lesssim 2^{-\ell(T')/n}$.

We are ready to show that for $T \in \mathcal{T} \in \mathbb{T}$, the difference in levels of any $K' \in \texttt{refine}(\mathcal{T}, \{T\})$ and its ancestor $K \in \mathcal{T}$ is uniformly bounded.

Theorem 3.3. *Let $T \in \mathcal{T} \in \mathbb{T}$ and $\mathcal{T}' = \texttt{refine}(\mathcal{T}, \{T\})$. Let $K \in \mathcal{T}$ and $K' \in \mathcal{T}'$ with $K' \subseteq K$. Then it holds that*

$$\ell(K') - \ell(K) \lesssim 1.$$

Proof. If $\ell(K') = \ell(K)$, the assertion is trivially valid. Hence, assume that $\ell(K) + 1 \le \ell(K')$, i.e., K' is newly created by the call. Recall the constant C from Lemma 1.1.

Case 1. If $\ell(T) \le \ell(K) + C$, then by Lemma 3.2 (a), it holds that $\ell(K') \le \ell(T) + 1 \le \ell(K) + C + 1$.

Case 2. If $\ell(T) > \ell(K) + C$, then Lemma 1.1 (b) implies that $\operatorname{dist}(T, K) \gtrsim 2^{-\ell(K)/n}$, whence

$$\operatorname{dist}(T, K') \gtrsim 2^{-\ell(K)/n}.$$

On the other hand, Lemma 3.2 (b) states that

$$\operatorname{dist}(K', T) \lesssim 2^{-\ell(K')/n}.$$

The foregoing two inequalities imply

$$2^{-\ell(K)/n} \lesssim 2^{-\ell(K')/n},$$

and so $\ell(K') - \ell(K) \lesssim 1$. □

Remark 3.4. In dimension $n = 2$, given $\mathcal{T} \in \mathbb{T}$, the triangulation $\mathcal{T}'$ defined by replacing each $T \in \mathcal{T}$ by its four grandchildren is conforming and so belongs to $\mathbb{T}$. We conclude that for any $T \in \mathcal{T}$, it holds that $\texttt{refine}(\mathcal{T}, \{T\}) \le \mathcal{T}'$ giving an easy proof of Theorem 3.3 in this case. Moreover, it yields the additional information that this theorem is valid in this situation with $\ell(K') - \ell(K) \le 2$.

This argument does not apply in $n > 2$ dimensions. Replacing any $T \in \mathcal{T} \in \mathbb{T}$ by its level n-descendants generally does not yield a conforming partition. Indeed, already for $n = 3$, in the partition formed by the level 3 descendants of a tagged tetrahedron T of type 0 or 1, all the edges of T have been cut exactly once, but for a tagged tetrahedron T of type 2, this partition still contains one of the original edges.

The following corollary generalizes Theorem 3.3 to the case that `refine` is called with a set of marked elements.

Corollary 3.5. *Let $\mathcal{M} \subseteq \mathcal{T} \in \mathbb{T}$ and $\mathcal{T}' = \texttt{refine}(\mathcal{T}, \mathcal{M})$. Let $K \in \mathcal{T}$ and $K' \in \mathcal{T}'$ with $K' \subseteq K$. Then it holds that*

$$\ell(K') - \ell(K) \lesssim 1.$$

Proof. It holds that

$$\mathcal{T}' = \bigvee_{T \in \mathcal{M}} \texttt{refine}(\mathcal{T}, \{T\}),$$

i.e., $\mathcal{T}'$ is the smallest common refinement of the triangulations $\texttt{refine}(\mathcal{T}, \{T\})$ for $T \in \mathcal{M}$. From Remark 3.1, we infer that for any $K' \in \mathcal{T}'$, there exists a $T \in \mathcal{M}$ with $K' \in \texttt{refine}(\mathcal{M}, \{T\})$. Thus, Theorem 3.3 proves the assertion. □

Remark 3.6. Corollary 3.5 accomplishes the proof of [2, Lemma 4.2]. It is furthermore required in [1, p. 1201] for the constant C_{son} in equation (2.8) of [1] to be finite.

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