SIAM J. NUMER. ANAL.
Vol. 59, No. 1, pp. 249–264

A POSTERIORI ERROR ANALYSIS OF THE INF-SUP CONSTANT FOR THE DIVERGENCE*

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Abstract. Two a posteriori error estimates for a numerical approximation scheme for the inf-sup constant for the divergence (also known as the LBB constant) are shown. Under the assumption that the inf-sup constant is an eigenvalue of the Cosserat operator separated from the essential spectrum and that the mesh size is sufficiently small, the first estimate bounds the eigenvalue and eigenfunction errors from above and below by an error estimator up to multiplicative constants. In the second error estimate the reliability constant converges to 1 as the mesh size decreases, at the expense of a suboptimal efficiency estimate, and so allows for guaranteed enclosures of the inf-sup constant on sufficiently fine meshes.

Key words. inf-sup constant, LBB constant, Stokes system, Cosserat spectrum, a posteriori error estimate, mixed finite element method

AMS subject classifications. 65N12, 65N15, 65N30, 76D07

DOI. 10.1137/20M1332529

1. Introduction.

The inf-sup constant β, sometimes also called the LBB constant, determines the continuity constant β^{-1} of a right-inverse to the divergence operator $\operatorname{div} : V \to Q$. In the L^2 setting with $\Omega \subseteq \mathbb{R}^n$ for $n \geq 2$ being an open, bounded, connected Lipschitz polytope, $V := H^1_0(\Omega; \mathbb{R}^n)$ is the space of vector-valued first-order Sobolev functions with zero boundary conditions and $Q := L^2_0(\Omega)$ denotes the L^2 functions with vanishing mean over Ω. The inf-sup constant is defined by

$$\beta := \inf_{q\in Q\setminus\{0\}} \frac{|\nabla q|_{-1}}{\|q\|_{L^2(\Omega)}} = \inf_{q\in Q\setminus\{0\}} \sup_{v\in V\setminus\{0\}} \frac{(q, \operatorname{div} v)_{L^2(\Omega)}}{\|q\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)}}. \tag{1.1}$$

Here, $|\nabla q|_{-1}$ denotes the norm in the dual of V and is given by the more explicit expression on the right-hand side of (1.1), ∇ is the gradient operator acting on scalar functions, and D denotes the derivative (matrix) of vector fields; more details on the notation can be found at the end of this section. The condition $\beta > 0$ is critical for stability considerations in fluid mechanics and elasticity theory [19, 13, 7]. It is well known [1] that, for a large class of domains, which in particular includes Lipschitz polytopes, the inf-sup constant is indeed positive $\beta > 0$. In this case there is an equivalent characterization of β^2 as the least nonzero element in the spectrum of the Cosserat operator $\Delta^{-1}\nabla \operatorname{div} : V \to V$, and the Rayleigh quotient reads

$$\beta^2 = \inf_{v\neq 0} \frac{\|\operatorname{div} v\|^2_{L^2(\Omega)}}{\|Dv\|^2_{L^2(\Omega)}}, \tag{1.2}$$

where the infimum is taken over the V-orthogonal complement of the divergence-free functions in V; here and throughout this work, V-orthogonality refers to orthogonality with respect to the scalar product $(D\cdot, D\cdot)_{L^2(\Omega)}$.

*Received by the editors April 17, 2020; accepted for publication (in revised form) October 2, 2020; published electronically January 25, 2021.
https://doi.org/10.1137/20M1332529

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It is immediate from the Rayleigh quotient representation that the underlying eigenvalue problem is noncompact, and so the usual tools from the numerical approximation of compact symmetric eigenvalue problems [2, 4] are not directly applicable. On nonsmooth domains, the spectrum of the Cosserat operator admits nontrivial essential parts [3, 11]. Moreover, even if β^2 is an isolated eigenvalue, the usual continuous-velocity finite element pairs (see., e.g., [5]) may have discrete inf-sup constants that do not converge to β. These and other remarkable properties as well as striking examples were provided and studied in [3, 8]. In view of these results, compatible discretizations are desirable. The numerical method from [12] provably produces monotonically decreasing approximations to β under mesh refinement and can thus be viewed as a Rayleigh–Ritz method, or more precisely as a variant thereof because the numerator in (1.1) is not computable (even for discrete quantities) and requires a further approximation step, namely the use of a discrete H^{-1} norm. It was shown in [12] that the approximation converges to β, and convergence rates were established for the case of β being an eigenvalue well separated from the essential spectrum. Experimental numerical results from [12] furthermore empirically showed reliability and efficiency of an a posteriori error estimator, which was derived using heuristic arguments. The aim of this paper is to theoretically justify this observation by complementing the a priori error analysis of [12] by a posteriori error estimates that relate the eigenvalue error to computationally accessible residuals. Under the assumption that the squared inf-sup constant $\mu = \beta^2$ is an isolated eigenvalue of the Cosserat operator, it is shown that the eigenvalue error is proportional to a computable quantity up to mesh-independent constants that depend on the spectral gap. That a posteriori quantity is the L^2 norm of the divergence-free part in a Helmholtz decomposition of the discrete eigenfunction, which is a discrete gradient, but not necessarily a gradient in a pointwise sense, and it can thus be seen as a nonconformity residual, which can be bounded by well-known residual-based error estimators or by a gradient reconstruction, which gives better control over the involved constants. A second reliability error estimate is furthermore shown that establishes an upper bound for the eigenvalue error with a multiplicative constant that is asymptotically equal to 1. This gives rise to an asymptotic lower eigenvalue bound, at the expense of a suboptimal efficiency estimate. The asymptotic character of the lower bound lies in the fact that, for the estimate to hold, the projection $\Pi\xi_h$ of the L^2-normalized discrete eigenfunction on the continuous eigenspace needs to satisfy the closeness assumption $b(\Pi\xi_h, \xi_h) \geq \lambda$ for the L^2 inner product b and some constant $0 < \lambda \leq 1$. This will usually require the mesh to be fine enough, which is difficult to quantify in practice. Both variants are practically computed in numerical experiments, where it is observed that the method gives convincing results even on coarse meshes that might not meet all assumptions of the theory.

Besides the method in [12] discussed above, there are only a few methods available for approximating the inf-sup constant. The first contribution to the convergence analysis for the inf-sup constant is [3], where sufficient conditions on continuous-velocity pairings were formulated which guarantee (plain) convergence of the corresponding inf-sup constants towards β. Another approach based on a least-squares formulation was presented in [17]. The results of this paper form the first contribution to a posteriori error control and two-sided enclosures of the inf-sup constant. Those are not only relevant to the Stokes equations, their stability analysis, and a posteriori error estimation [14, 15], but also to precise bounds for Korn's constant in elasticity [7]. The difference of the present error analysis with respect to existing works on a posteriori error bounds for eigenvalue problems is that, in the absence of a compact embedding, the eigenvalue problem is not a mere perturbation of the related linear problem by a higher-order term. Instead, a careful choice of the discrete setting is needed to grant compatibility of the numerical scheme.

The remaining parts of this article are organized as follows: section 2 provides the formulation of the eigenvalue problem related to the inf-sup constant and presents a short review of the numerical method of [12]. The lemmas of section 3 provide several identities that are used in the a posteriori error analysis of section 4. The error bounds are empirically investigated in the numerical experiments of section 5.

Standard notation on Lebesgue and Sobolev spaces applies throughout this paper. Given an open domain $\omega \subseteq \Omega$, the L^2 inner product and norm are denoted by $(\cdot,\cdot)_{L^2(\omega)}$ and $\|\cdot\|_{L^2(\omega)}$; the spaces of scalar, vector-valued, and matrix-valued L^2 functions over ω are denoted by $L^2(\omega)$, $L^2(\omega;\mathbb{R}^n)$, and $L^2(\omega;\mathbb{R}^{n\times n})$, respectively; the subspace of $L^2(\Omega)$ of functions with vanishing global average is denoted by $Q = L^2_0(\Omega)$. The derivative (matrix) of a vector-valued function v is denoted by Dv, the trace of a square matrix A is denoted by $\operatorname{tr} A$, and the divergence of a vector field ϕ is denoted by $\operatorname{div}\phi$ and satisfies $\operatorname{div}\phi = \operatorname{tr} D\phi$ provided $D\phi$ exists. The gradient of a scalar-valued function v reads ∇v, and the Laplacian of a (scalar- or vector-valued) function v reads Δv. The space $V = H^1_0(\Omega;\mathbb{R}^n)$ is equipped with the energy inner product $(D\cdot, D\cdot)_{L^2(\Omega)}$ and its norm $|\cdot|_1$, and its dual space is endowed with the corresponding dual norm $|\cdot|_{-1}$ as in (1.1). The space of L^2 vector fields with divergence in $L^2(\Omega)$ is denoted by $H(\operatorname{div},\Omega)$.

2. Statement of the problem and review of the numerical scheme. On $L^2(\Omega;\mathbb{R}^{n\times n})$ define the following two bilinear forms:

$$a(\sigma,\tau) := (\operatorname{tr}\sigma, \operatorname{tr}\tau)_{L^2(\Omega)} \quad \text{and} \quad b(\sigma,\tau) := (\sigma,\tau)_{L^2(\Omega)} \quad \text{for any } \sigma,\tau \in L^2(\Omega;\mathbb{R}^{n\times n}).$$

The induced (semi)norms are denoted by

$$\|\tau\|_a := \sqrt{a(\tau,\tau)} \quad \text{and} \quad \|\tau\|_b := \sqrt{b(\tau,\tau)} \quad \text{for all } \tau \in L^2(\Omega;\mathbb{R}^{n\times n}).$$

The subspace of divergence-free elements in V is denoted by $Z := \{v \in V : \operatorname{div} v = 0\}$, and its V-orthogonal complement is denoted by $Z^{\perp_1}$. Given the space $\Gamma := DV$ of vector gradients, the space $X := D(Z^{\perp_1})$ is characterized as

$$X = \{\tau \in \Gamma : b(\tau,\eta) = 0 \text{ for all } \eta \in \Gamma \text{ with } \operatorname{tr}\eta = 0\}.$$

With this notation, the squared inf-sup constant $\mu := \beta^2$ from (1.2) reads

$$\mu = \inf_{\tau \in X\setminus\{0\}} \frac{\|\tau\|_a^2}{\|\tau\|_b^2}. \tag{2.1}$$

The eigenvalue problem related to the Rayleigh quotient (2.1) is to find a nonzero eigenfunction $\xi \in X$ and an eigenvalue $\xi \in \mathbb{R}$ such that

$$a(\xi,\tau) = \mu b(\xi,\tau) \quad \text{for all } \tau \in X. \tag{2.2}$$

In case that the bottom of the spectrum of the Cosserat operator is an eigenvalue, the minimum (2.1) is attained and the minimizer satisfies (2.2) with the same number μ. It is unknown in general whether the infimum in (2.1) is actually a minimum, and this property will depend on the domain. The theory in this paper is based on the assumption that the minimum is achieved and that μ is an isolated eigenvalue. This property is stated as Assumption A in section 4 below.

The discretization from [12] is based on a space Γ_h of so-called discrete gradients, which are not gradients in a pointwise sense. The space was introduced in [16] where it was used to generalize the nonconforming P_1 finite element [6] (Crouzeix–Raviart method) to higher polynomial degrees. It is based on the implicit definition of discrete gradients as the orthogonal complement of divergence-free objects within the space of piecewise polynomial vector fields with respect to a regular simplicial partition $\mathcal{T}$ of the domain Ω. Given a fixed polynomial degree $k \geq 0$, the space of polynomials with respect to a domain $\omega \subseteq \mathbb{R}^n$ of degree not larger than k with values in $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ is denoted by $P_k(\omega)$, $P_k(\omega;\mathbb{R}^n)$, $P_k(\omega;\mathbb{R}^{n\times n})$, respectively. The L^2 functions over Ω that are piecewise polynomials with respect to $\mathcal{T}$ are analogously denoted by $P_k(\mathcal{T})$, $P_k(\mathcal{T};\mathbb{R}^n)$, $P_k(\mathcal{T};\mathbb{R}^{n\times n})$. The Raviart–Thomas finite element space [5] is defined by

$$RT_k(\mathcal{T}) := \left\{ q \in H(\operatorname{div},\Omega) \,\middle|\, \begin{array}{l} \forall T \in \mathcal{T}\ \exists(\alpha,\beta) \in P_k(T;\mathbb{R}^n) \times P_k(T), \\ \forall x \in T\ q|_T(x) = \alpha(x) + \beta(x)x \end{array} \right\}.$$

It is well known and follows from the definition of the weak derivative that the space Γ is the L^2-orthogonal complement of the space $\mathfrak{Z} := \{v \in H(\operatorname{div},\Omega)^n : \operatorname{div} v = 0\}$ (where div acts row-wise on matrix fields), written $\Gamma = \mathfrak{Z}^{\perp_{L^2}}$. The definition of Γ_h is given by the following discrete analogue. Let $\mathfrak{Z}_h := RT_k(\mathcal{T})^n \cap \mathfrak{Z}$ denote the space of divergence-free Raviart–Thomas fields. It is known [10, Lemma 3.1] that $\mathfrak{Z}_h$ is a subset of $P_k(\mathcal{T};\mathbb{R}^{n\times n})$. The space Γ_h of discrete gradients is then defined as the L^2-orthogonal complement of $\mathfrak{Z}_h$ within $P_k(\mathcal{T};\mathbb{R}^{n\times n})$, written

$$\Gamma_h := \mathfrak{Z}_h^{\perp_{L^2}} \subseteq P_k(\mathcal{T};\mathbb{R}^{n\times n}).$$

It is important to note that Γ_h may contain functions that are not pointwise gradients, i.e., $\Gamma_h \not\subseteq \Gamma$. Let $\Pi_h : L^2(\Omega) \to P_k(\mathcal{T})$ denote the L^2 projection onto the piecewise polynomials of degree k. If applied to tensors, the action of Π_h is understood componentwise. The discrete gradients satisfy the following projection property [12]:

$$\Pi_h \Gamma \subseteq \Gamma_h. \tag{2.3}$$

The discrete analogue to the space X reads

$$X_h := \{\tau_h \in \Gamma_h : b(\tau_h,\eta_h) = 0 \text{ for all } \eta_h \in \Gamma_h \text{ with } \operatorname{tr}\eta_h = 0\}.$$

Again, it is to be expected that $X_h \not\subseteq X$ is a nonconforming approximation. The orthogonal projection to the subspace X_h is denoted by $\mathfrak{P}_h$. It was shown in [12, Lemma 7] that $\mathfrak{P}_h$ preserves the trace in the following sense:

$$\operatorname{tr}(\mathfrak{P}_h\gamma) = \operatorname{tr}(\Pi_h\gamma) \quad \text{for any } \gamma \in \Gamma. \tag{2.4}$$

The numerical approximation μ_h to μ from (2.1) is defined by the discrete Rayleigh quotient

$$\mu_h = \inf_{\tau_h \in X_h\setminus\{0\}} \frac{\|\tau_h\|_a^2}{\|\tau_h\|_b^2}.$$

In the discrete setting, this is equivalent to finding the first eigenpair $(\mu_h,\xi_h) \in \mathbb{R}\times X_h$ with $\|\xi_h\|_b = 1$ to the discrete Cosserat eigenvalue problem

$$a(\xi_h,\tau_h) = \mu_h b(\xi_h,\tau_h) \quad \text{for all } \tau_h \in X_h. \tag{2.5}$$

In what follows, the choice of the normalized discrete eigenfunction ξ_h defining the first eigenpair (also in the case of nontrivial multiplicity) is arbitrary. It was shown in [12] that, alternatively, $\beta_h := \mu_h^{1/2}$ is characterized in the format of an inf-sup constant as in (1.1). This is based on a conforming pressure discretization $Q_h \subseteq Q$ with $Q_h := P_k(\mathcal{T}) \cap Q$ and a (computable) discrete version of the H^{-1} norm of any ∇q_h with $q_h \in Q_h$, which is defined by

$$|\nabla q_h|_{-1,h} := \sup_{\gamma_h \in \Gamma_h \setminus \{0\}} \frac{(q_h, \operatorname{tr} \gamma_h)_{L^2(\Omega)}}{\|\gamma_h\|_{L^2(\Omega)}}.$$

The discrete inf-sup constant is then given by

$$\beta_h = \inf_{p_h \in Q_h \setminus \{0\}} \frac{|\nabla p_h|_{-1,h}}{\|p_h\|_{L^2(\Omega)}} \tag{2.6}$$

and $q_h \in Q_h$ minimizes (2.6) if and only if

$$q_h = \operatorname{tr} \xi_h \tag{2.7}$$

for some nonzero eigenfunction ξ_h of (2.5); see [12, Proposition 6]. The discrete norm possesses the monotonicity property

$$|\nabla q_h|_{-1} \leq |\nabla q_h|_{-1,h} \quad \text{for any } q_h \in Q_h, \tag{2.8}$$

and, accordingly, the approximations β_h converge to β monotonically from above under shape-regular mesh refinement [12, Theorem 2.1]. Provided that β^2 is a separated eigenvalue with an eigenfunction u of class $H^{1+s}(\Omega)$, the eigenvalue error decreases at rate $2\min\{k+1, s\}$ with respect to the maximum mesh size [12].

3. Preparatory identities. This section is devoted to the proofs of several lemmas that provide identities for discrete eigenpairs. The first lemma states that discrete eigenpairs may be tested against continuous quantities.

LEMMA 3.1 (consistency of discrete eigenpairs). *Any discrete eigenpair* $(\mu_h, \xi_h) \in \mathbb{R} \times X_h$ *of* (2.5) *satisfies*

$$a(\xi_h, \gamma) = \mu_h b(\xi_h, \gamma) \quad \textit{for all } \gamma \in \Gamma.$$

Proof. The definition of Π_h, property (2.4) of the projection $\mathfrak{P}_h$, and the discrete eigenvalue problem (2.5) yield

$$a(\xi_h, \gamma) = (\operatorname{tr} \xi_h, \Pi_h \operatorname{tr} \gamma)_{L^2(\Omega)} = (\operatorname{tr} \xi_h, \operatorname{tr} \mathfrak{P}_h \gamma)_{L^2(\Omega)} = a(\xi_h, \mathfrak{P}_h \gamma) = \mu_h b(\xi_h, \mathfrak{P}_h \gamma).$$

Relation (2.3) states that $\Pi_h \gamma \in \Gamma_h$, and identity (2.4) implies $\operatorname{tr}(\mathfrak{P}_h - \Pi_h)\gamma = 0$, whence $(\mathfrak{P}_h - \Pi_h)\gamma$ is b-orthogonal to any element of X_h. Therefore,

$$\mu_h b(\xi_h, \mathfrak{P}_h \gamma) = \mu_h b(\xi_h, (\mathfrak{P}_h - \Pi_h)\gamma) + \mu_h b(\xi_h, \Pi_h \gamma) = \mu_h b(\xi_h, \gamma).$$

The combination with the first chain of identities proves the assertion. ∎

LEMMA 3.2. *Let* $(\mu_h, \xi_h) \in \mathbb{R} \times X_h$ *be a discrete eigenpair of* (2.5) *with* $\|\xi_h\|_b = 1$, *and let* $\hat{\xi} \in \Gamma \setminus \{0\}$ *with* $\hat{\mu} := \|\hat{\xi}\|_a^2 / \|\hat{\xi}\|_b^2$. *Then*

$$\mu_h - \hat{\mu} + \|\xi_h - \hat{\xi}\|_a^2 = \mu_h \|\xi_h - \hat{\xi}\|_b^2 + (\mu_h - \hat{\mu})(1 - \|\hat{\xi}\|_b^2).$$

Proof. The normalization of ξ_h, the definition of $\hat{\mu}$, and elementary computations yield

$$
\begin{aligned}
\mu_h - \hat{\mu} &= \|\xi_h\|_a^2 - \frac{\|\hat{\xi}\|_a^2}{\|\hat{\xi}\|_b^2} \\
&= \|\xi_h\|_a^2 - \|\hat{\xi}\|_a^2 + \frac{\|\hat{\xi}\|_b^2 - 1}{\|\hat{\xi}\|_b^2}\|\hat{\xi}\|_a^2 = \|\xi_h\|_a^2 - \|\hat{\xi}\|_a^2 - (1 - \|\hat{\xi}\|_b^2)\hat{\mu}.
\end{aligned} \tag{3.1}
$$

Elementary algebraic manipulations with the symmetric form a reveal for the first two terms on the right-hand side

$$
\|\xi_h\|_a^2 - \|\hat{\xi}\|_a^2 = a(\xi_h + \hat{\xi}, \xi_h - \hat{\xi}) = 2a(\xi_h, \xi_h - \hat{\xi}) - \|\xi_h - \hat{\xi}\|_a^2. \tag{3.2}
$$

The discrete eigenvalue problem (2.5) and Lemma 3.1 together with direct computations based on $\|\xi_h\|_b = 1$ show that

$$
\begin{aligned}
2a(\xi_h, \xi_h - \hat{\xi}) = 2\mu_h b(\xi_h, \xi_h - \hat{\xi}) &= \mu_h(1 - 2b(\xi_h, \hat{\xi}) + \|\hat{\xi}\|_b^2) + \mu_h(1 - \|\hat{\xi}\|_b^2) \\
&= \mu_h\|\xi_h - \hat{\xi}\|_b^2 + \mu_h(1 - \|\hat{\xi}\|_b^2).
\end{aligned} \tag{3.3}
$$

The combination of (3.2) and (3.3) results in

$$
\|\xi_h\|_a^2 - \|\hat{\xi}\|_a^2 = \mu_h\|\xi_h - \hat{\xi}\|_b^2 + \mu_h(1 - \|\hat{\xi}\|_b^2) - \|\xi_h - \hat{\xi}\|_a^2.
$$

Inserting this identity on the right-hand side of (3.1) yields the assertion. ∎

The following results relate error quantities to a residual. Given a pair $(\hat{\mu}, \hat{\xi}) \in \mathbb{R} \times \Gamma$, the residual evaluated at $\xi_h \in X_h$ is defined by

$$
\operatorname{Res}_{\hat{\mu},\hat{\xi}}(\xi_h) := \hat{\mu} b(\hat{\xi}, \xi_h) - a(\hat{\xi}, \xi_h). \tag{3.4}
$$

The following lemma states a residual identity for the eigenvalue approximation error.

LEMMA 3.3. *Let $(\mu_h, \xi_h) \in \mathbb{R} \times X_h$ be a discrete eigenpair of* (2.5)*, and let $\hat{\xi} \in \Gamma$ and $\hat{\mu} \in \mathbb{R}$. Then*

$$
(\mu_h - \hat{\mu})b(\hat{\xi}, \xi_h) = -\operatorname{Res}_{\hat{\mu},\hat{\xi}}(\xi_h).
$$

Proof. The symmetry of b and Lemma 3.1 show that

$$
(\mu_h - \hat{\mu})b(\hat{\xi}, \xi_h) = \mu_h b(\xi_h, \hat{\xi}) - \hat{\mu} b(\hat{\xi}, \xi_h) = a(\xi_h, \hat{\xi}) - \hat{\mu} b(\hat{\xi}, \xi_h).
$$

The symmetry of a and the definition in (3.4) show that this equals $-\operatorname{Res}_{\hat{\mu},\hat{\xi}}(\xi_h)$. ∎

By the L^2-orthogonal decomposition (Helmholtz decomposition) into a gradient and a divergence-free part, there exist $\alpha \in \Gamma$ and $R \in L^2(\Omega; \mathbb{R}^n)$ with $\operatorname{div} R = 0$ (that is, $R \in \mathfrak{Z}$ in the previous notation) such that ξ_h is decomposed as

$$
\xi_h = \alpha + R. \tag{3.5}
$$

In this decomposition, $\alpha \in \Gamma$ is the L^2 projection of ξ_h on Γ, and R is the remainder accounting for the expected nonconformity $\xi_h \notin \Gamma$. The following identities relate the residual of an exact eigenpair to expressions containing R.

$$\begin{tikzcd}
Z^{\perp_1} \arrow[dr, "\operatorname{div}"] & & & X_h \arrow[dl, "\operatorname{tr}"'] \\
& L^2_0(\Omega) = Q \arrow[dl, "\nabla"] & Q_h \arrow[l, hook'] \arrow[dr, "\nabla"] & \\
(Z^{\perp_1})^* \arrow[uu, "\Delta^{-1}"] & & & (X_h)^*
\end{tikzcd}$$

FIG. 1. *Overview of the relevant spaces and operators.*

LEMMA 3.4. *Let $(\mu_h, \xi_h) \in \mathbb{R} \times X_h$ be a discrete eigenpair of (2.5) with $\|\xi_h\|_b = 1$, and let $(\mu, \xi) \in \mathbb{R} \times \Gamma$ be an eigenpair of (2.2). Then*

$$-\operatorname{Res}_{\mu,\xi}(\xi_h) = a(\xi, R) = a(\xi - \xi_h, R) + \mu_h \|R\|_b^2.$$

Proof. The definition of the residual (3.4) and the decomposition (3.5) yield

$$-\operatorname{Res}_{\mu,\xi}(\xi_h) = a(\xi, \alpha + R) - \mu b(\xi, \alpha + R) = a(\xi, \alpha) + a(\xi, R) - \mu b(\xi, \alpha), \tag{3.6}$$

where it was used that R is b-orthogonal to ξ. The field α admits an L^2-orthogonal decomposition into a trace-free vector gradient Dz with $z \in V$ and a gradient field $\sigma \in X$ so that $\alpha = Dz + \sigma$. Since $\xi \in X$ is b-orthogonal to Dz and Dz is trace-free, the eigenvalue problem (2.2) shows that

$$a(\xi, \alpha) - \mu b(\xi, \alpha) = a(\xi, Dz) + a(\xi, \sigma) - \mu b(\xi, Dz) - \mu b(\xi, \sigma) = a(\xi, \sigma) - \mu b(\xi, \sigma) = 0.$$

The combination of this with (3.6) establishes the first asserted identity. It furthermore follows from elementary manipulations with (3.5) that

$$a(\xi, R) = a(\xi - \xi_h, R) + a(\xi_h, \xi_h) - a(\xi_h, \alpha).$$

The discrete eigenvalue problem (2.5) and Lemma 3.1 together with the b-orthogonal decomposition (3.5) show that this equals

$$a(\xi - \xi_h, R) + \mu_h b(\xi_h, R) = a(\xi - \xi_h, R) + \mu_h \|R\|_b^2,$$

which establishes the second claimed identity. ∎

4. A posteriori error analysis.

The relevant operators used in the error analysis are illustrated in the diagram of Figure 1. It is well known that the Cosserat operator $\Delta^{-1}\nabla \operatorname{div} : Z^{\perp_1} \to Z^{\perp_1}$ and the Schur complement of the Stokes system $\operatorname{div} \Delta^{-1}\nabla : L^2_0(\Omega) \to L^2_0(\Omega)$ have the same spectrum. The usefulness of this observation lies in the fact that $Q_h \subseteq Q$ is a conforming approximation.

LEMMA 4.1. *The spectra of the Cosserat operator $\mathcal{C} := \Delta^{-1}\nabla \operatorname{div} : Z^{\perp_1} \to Z^{\perp_1}$ and the Schur complement of the Stokes system $S := \operatorname{div} \Delta^{-1}\nabla : L^2_0(\Omega) \to L^2_0(\Omega)$ are identical. A function $u \in Z^{\perp_1}$ is an eigenfunction with eigenvalue μ of $\mathcal{C}$ if and only if $\operatorname{div} u$ is an eigenfunction of S corresponding to μ.*

Proof. A proof is given for convenient reading. Since $\operatorname{div} : Z^{\perp_1} \to L^2_0(\Omega)$ is an isomorphism, and so is its negative adjoint $\nabla : L^2_0(\Omega) \to (Z^{\perp_1})^*$, the claim on the spectra is a consequence of the diagram of Figure 1. Indeed, $(\mathcal{C} - \mu) = (\Delta^{-1}\nabla \operatorname{div} - \mu)$ is invertible if and only if $(\Delta^{-1}\nabla - \mu \operatorname{div}^{-1})$ is invertible (multiply with div^{-1} from the right) if and only if $(\operatorname{div}\Delta^{-1}\nabla - \mu) - \mu = (S - \mu)$ is invertible (multiply with div from the left). If (μ, u) is an eigenpair of $\mathcal{C}$, then $\operatorname{div} u$ belongs to $L^2_0(\Omega)$. Applying the div operator to $\mathcal{C}u = \mu u$ shows $\operatorname{div}\Delta^{-1}\nabla(\operatorname{div} u) = \mu(\operatorname{div} u)$, which is the eigenvalue relation for S. Conversely, if (μ, q) is an eigenpair of S, the inf-sup condition shows that $q = \operatorname{div} u$ for some $u \in Z^{\perp_1}$. Thus, $\operatorname{div}\Delta^{-1}\nabla(\operatorname{div} u) = \mu(\operatorname{div} u)$, and applying div^{-1} yields the eigenvalue relation for $\mathcal{C}$. ∎

The error analysis of this paper makes use of the following separation assumption.

ASSUMPTION A. *There exists a minimizing function τ in* (2.1) *and μ is an isolated eigenvalue of the Cosserat operator in the sense that there exists $\delta > 0$ such that $(\mu, \mu + \delta] \cap \sigma = \emptyset$, where σ denotes the Cosserat spectrum.*

Note that Assumption A does not require that μ be a simple eigenvalue.

LEMMA 4.2. *Let $(\mu_h, \xi_h) \in \mathbb{R} \times X_h$ be the first eigenpair of* (2.5) *with $q_h := \operatorname{tr}\xi_h$. Let Assumption A hold, and let μ denote the least eigenvalue of* (2.2). *Let $W \subseteq Z^{\perp_1}$ denote the eigenspace related to μ, and denote $M := \{\operatorname{tr}\sigma : \sigma \in W\}$. Let P be the L^2 projection onto M. Then,*

$$\delta\|(1 - P)q_h\|^2_{L^2(\Omega)} \le (\mu_h - \mu)\|q_h\|^2_{L^2(\Omega)} = (\mu_h - \mu)\mu_h\|\xi_h\|^2_b.$$

Proof. Parts of the proof follow the ideas of [3, Theorem 5.3]. The spectral gap property [20, section 4.3] and the self-adjointness of S imply

$$\begin{aligned}(\mu + \delta)\|(1 - P)q_h\|^2_{L^2(\Omega)} &\le (S(1 - P)q_h, (1 - P)q_h)_{L^2(\Omega)}\\ &= (SPq_h, Pq_h)_{L^2(\Omega)} - 2(Sq_h, Pq_h)_{L^2(\Omega)} + (Sq_h, q_h)_{L^2(\Omega)}.\end{aligned}$$

Since Pq_h is an eigenfunction of S with eigenvalue μ, and S is self-adjoint, the sum of the first two terms on the right-hand side equals

$$\mu(Pq_h, Pq_h)_{L^2(\Omega)} - 2\mu(q_h, Pq_h)_{L^2(\Omega)} = -\mu\|Pq_h\|^2_{L^2(\Omega)},$$

while the last term satisfies, due to the definition of S and the isometry property of the Laplacian,

$$(Sq_h, q_h)_{L^2(\Omega)} = -(\Delta^{-1}\nabla q_h, \nabla q_h)_{L^2(\Omega)} = |\Delta^{-1}\nabla q_h|^2_1 = |\nabla q_h|^2_{-1}.$$

Altogether,

$$(\mu + \delta)\|(1 - P)q_h\|^2_{L^2(\Omega)} \le |\nabla q_h|^2_{-1} - \mu\|Pq_h\|^2_{L^2(\Omega)}.$$

The monotonicity (2.8) and relation (2.7) imply

$$|\nabla q_h|_{-1} \le |\nabla q_h|_{-1,h} = \mu_h^{1/2}\|q_h\|_{L^2(\Omega)},$$

so that

$$(\mu + \delta)\|(1 - P)q_h\|^2_{L^2(\Omega)} \le \mu_h\|q_h\|^2_{L^2(\Omega)} - \mu\|Pq_h\|^2_{L^2(\Omega)}.$$

After expanding $\mu\|(1-P)q_h\|^2_{L^2(\Omega)} = \mu\|q_h\|^2_{L^2(\Omega)} - \mu\|Pq_h\|^2_{L^2(\Omega)}$ one thus finds that

$$\delta\|(1-P)q_h\|^2_{L^2(\Omega)} \le (\mu_h - \mu)\|q_h\|^2_{L^2(\Omega)},$$

which, together with $\|q_h\|^2_{L^2(\Omega)} = \|\xi_h\|^2_a = \mu_h\|\xi_h\|^2_b$, proves the lemma. ∎

Under Assumption A, the eigenfunctions related to the smallest eigenvalue of (2.2) form a finite-dimensional subspace $W \subseteq Z^{\perp_1}$. The a-orthogonal projection on W is denoted by Π; that is, given any $\tau \in L^2(\Omega;\mathbb{R}^{n\times n})$, the element $\Pi\tau \in W$ satisfies

$$a(\Pi\tau,\sigma) = a(\tau,\sigma) \qquad \text{for all } \sigma \in W.$$

Since, by Lemma 4.1, $M = \{\operatorname{tr}\sigma : \sigma \in W\}$ is the eigenspace of S with respect to μ, it follows that

$$a(\Pi\tau,\sigma) = a(\tau,\sigma) = (\operatorname{tr}\tau, \operatorname{tr}\sigma)_{L^2(\Omega)} = (P\operatorname{tr}\tau, \operatorname{tr}\sigma)_{L^2(\Omega)} \quad \text{for any } \sigma \in W.$$

This proves the relation

$$\operatorname{tr}\Pi\tau = P\operatorname{tr}\tau \quad \text{for all } \tau \in L^2(\Omega;\mathbb{R}^{n\times n}). \tag{4.1}$$

In what follows, the discrete spaces are related to a sequence of simplicial meshes (labeled by the index h) within a shape-regular family whose maximum mesh size converges to 0, written $h \to 0$. The next lemma ensures that, under Assumption A, the critical constant $b(\Pi\xi_h, \xi_h)$ converges to 1 under mesh refinement.

Lemma 4.3. *Let $(\mu_h, \xi_h) \in \mathbb{R} \times X_h$ be the first b-normalized eigenpair of* (2.5). *Assumption A implies*

$$b(\Pi\xi_h, \xi_h) \to 1 \quad \text{as } h \to 0.$$

Proof. Lemma 3.1, the symmetry of a and b, the projection property of Π, and the Pythagorean identity with $\mu_h = \|\xi_h\|^2_a$ imply

$$b(\Pi\xi_h, \xi_h) = \frac{a(\Pi\xi_h, \xi_h)}{\mu_h} = \frac{\|\Pi\xi_h\|^2_a}{\mu_h} = \frac{\mu_h - \|(1-\Pi)\xi_h\|^2_a}{\mu_h} = 1 - \frac{\|(1-\Pi)\xi_h\|^2_a}{\mu_h}.$$

This implies in particular that $b(\Pi\xi_h, \xi_h) \le 1$. Lemma 4.2 and identity (4.1) show that

$$\frac{\|(1-\Pi)\xi_h\|^2_a}{\mu_h} \le \frac{\mu_h - \mu}{\delta}.$$

The combination of the foregoing two displayed formulas results in

$$1 - \frac{\mu_h - \mu}{\delta} \le 1 - \frac{\|(1-\Pi)\xi_h\|^2_a}{\mu_h} \le b(\Pi\xi_h, \xi_h) \le 1,$$

and the stated convergence follows from the convergence $\mu_h \searrow \mu$ as $h \to 0$, which was established in [12]. ∎

Assumption B. *The mesh size is so small that there exists $\lambda > 0$ such that the first b-normalized eigenpair $(\mu_h, \xi_h) \in \mathbb{R} \times X_h$ of* (2.5) *satisfies*

$$\lambda \le b(\Pi\xi_h, \xi_h) \le 1$$

for all meshes in the shape-regular family of possible refinements.

The following result states upper bounds of the eigenvalue error in terms of information on R. Examples for two-sided computable bounds on the remaining quantity $\|R\|_b$ are well known and commented on in section 5.

THEOREM 4.4 (reliability I). *Let Assumptions A and B hold. Let* $(\mu_h, \xi_h) \in \mathbb{R} \times X_h$ *denote the first b-normalized eigenpair of* (2.5)*, and let* μ *denote the least eigenvalue of* (2.2)*. Then, for any positive* $0 < \varepsilon < \infty$*,*

$$(\mu_h - \mu)\left[b(\Pi\xi_h, \xi_h) - \frac{\varepsilon}{2}\right] \leq \mu_h \left[\frac{n}{2\varepsilon\delta} + 1\right] \|R\|_b^2.$$

Proof. The combination of Lemma 3.3 and Lemma 3.4 with $\xi := \Pi\xi_h$ yields

$$(\mu_h - \mu) b(\Pi\xi_h, \xi_h) = a(\Pi\xi_h - \xi_h, R) + \mu_h \|R\|_b^2. \tag{4.2}$$

The property (4.1) together with the Cauchy inequality and Lemma 4.2 show for the first term on the right-hand side of (4.2) that

$$a(\Pi\xi_h - \xi_h, R) \leq \|(1 - P)\operatorname{tr}\xi_h\|_{L^2(\Omega)} \|R\|_a \leq \sqrt{\mu_h}\sqrt{\frac{\mu_h - \mu}{\delta}}\|R\|_a.$$

This proves

$$(\mu_h - \mu) b(\Pi\xi_h, \xi_h) \leq \sqrt{\mu_h}\sqrt{\frac{\mu_h - \mu}{\delta}}\|R\|_a + \mu_h\|R\|_b^2.$$

The asserted estimate thus follows from Young's inequality

$$\sqrt{\mu_h}\sqrt{\frac{\mu_h - \mu}{\delta}}\|R\|_a + \mu_h\|R\|_b^2 \leq \varepsilon\frac{\mu_h - \mu}{2} + \mu_h\frac{\|R\|_a^2}{2\varepsilon\delta} + \mu_h\|R\|_b^2$$

and Hölder's inequality $\|R\|_a^2 \leq n\|R\|_b^2$ for finite sums. ∎

The following theorem states efficiency of $\|R\|_b^2$.

THEOREM 4.5 (efficiency). *Let Assumptions A and B hold. Let* $(\mu_h, \xi_h) \in \mathbb{R} \times X_h$ *denote the first b-normalized eigenpair of* (2.5)*, and let* μ *denote the least eigenvalue of* (2.2)*. Then,*

$$\|R\|_b^2 \leq (\mu_h - \mu)\left(\frac{1}{\mu} + \frac{1}{\delta}\right).$$

Proof. From the orthogonality of the Helmholtz decomposition (3.5), it follows that

$$\|R\|_b^2 \leq \|\alpha - \Pi\xi_h\|_b^2 + \|R\|_b^2 = \|\alpha - \Pi\xi_h + R\|_b^2 = \|(1 - \Pi)\xi_h\|_b^2,$$

which proves efficiency for the eigenfunction approximation.

The application of Lemma 3.2 with $\hat{\xi} := \Pi\xi_h$, identity (4.1), and Lemma 4.2 furthermore shows

$$\mu_h\|(1 - \Pi)\xi_h\|_b^2 = \|\xi_h - \Pi\xi_h\|_a^2 + \|\Pi\xi_h\|_b^2(\mu_h - \mu) \leq \frac{\mu_h - \mu}{\delta}\mu_h + \|\Pi\xi_h\|_b^2(\mu_h - \mu).$$

Furthermore, from the Rayleigh quotient for μ and the nonexpansivity $\|\Pi\xi_h\|_a^2 \leq \|\xi_h\|_a^2 = \mu_h$, it follows that

$$\|\Pi\xi_h\|_b^2 = \mu^{-1}\|\Pi\xi_h\|_a^2 \leq \mu_h/\mu.$$

The combination of the above estimates concludes the proof. ∎

The following result provides a reliability bound which allows for more precise control of the involved constants, at the expense of suboptimal efficiency.

THEOREM 4.6 (reliability II). *Let Assumptions A and B hold. Let* $(\mu_h, \xi_h) \in \mathbb{R} \times X_h$ *denote the first b-normalized eigenpair of* (2.5), *and let* μ *denote the least eigenvalue of* (2.2). *Then,*

$$\lambda(\mu_h - \mu) \leq (\mu_h - \mu) b(\Pi\xi_h, \xi_h) \leq \sqrt{\mu_h}\|R\|_a \leq \sqrt{n\mu_h}\|R\|_b.$$

Proof. Lemma 3.3 with $\hat{\xi} := \Pi\xi_h$ and Lemma 3.4 imply

$$(\mu_h - \mu) b(\Pi\xi_h, \xi_h) = -\operatorname{Res}_{\mu,\Pi\xi_h}(\xi_h) = a(\Pi\xi_h, R).$$

The assertion of the theorem then follows from the Cauchy inequality and the bounds $\|\Pi\xi_h\|_a \leq \|\xi_h\|_a = \sqrt{\mu_h}$ and $\|R\|_a \leq \sqrt{n}\|R\|_b$. ∎

A comparison of Theorem 4.6 with Theorem 4.5 reveals that the reliable bound of Theorem 4.6 is not efficient in the sense that the upper bound is expected to rather behave like the square root of the eigenvalue error. On the other hand, the bound of Theorem 4.6 is close to being explicit, and the remaining unknown constant $b(\Pi\xi_h, \xi_h)$ converges to 1 under mesh refinement, as shown in Lemma 4.3. If the mesh size is small enough, the eigenvalue error is bounded from above by $\sqrt{n}\mu_h\|R\|_b/\lambda$, and a computable bound thus requires an estimate on λ (which is generally not available) and a computable upper bound on $\|R\|_b$, which is not difficult to achieve, as shown in section 5.

5. Practical error estimators and numerical results.

This section presents practical bounds for the quantity $\|R\|_b$ and numerical experiments on adaptive meshes.

5.1. Error estimator. Reliable and efficient bounds for $\|R\|_b$ have been established in the literature; see [21, section 4.12] for the lowest-order case and [16] for arbitrary polynomial degree. In the two-dimensional case $n = 2$, given a triangle $T \in \mathcal{T}$ of the triangulation $\mathcal{T}$ with diameter h_T and set of edges $\mathcal{F}(T)$, the local contribution to the error estimator is defined by

$$\eta_{\mathrm{res}}^2(T) := h_T^2\|\operatorname{rot}\xi_h\|_{L^2(T)}^2 + h_T \sum_{F\in\mathcal{F}(T)} \|[\xi_h]_F t_F\|_{L^2(F)}^2. \tag{5.1}$$

Here, $\operatorname{rot}\xi_h = \partial_1(\xi_h)_2 - \partial_2(\xi_h)_1$, t_F is a unit tangent vector to the edge F, and the brackets $[\cdot]_F$ denote the jump across F for any interior edge F and the trace for any boundary edge. It is known [16] that

$$c\|R\|_b \leq \eta_{\mathrm{res}} \leq C\|R\|_b \quad \text{with } \eta_{\mathrm{res}} := \sqrt{\sum_{T\in\mathcal{T}} \eta_{\mathrm{res}}^2(T)}$$

with mesh-independent positive constants c, C, provided the domain is simply connected. Analogous estimators can be derived in higher space dimensions. This is a typical residual-based error estimator, and its effect on adaptive mesh refinement was already tested and documented in [12], where the error estimator had been derived on a heuristic level.

The results presented in this section shall shed light on the dependence of the reliability and efficiency estimates and the respective constants in the presence of coarse meshes or a narrow spectral gap. For better control over the reliability constant (which is relevant for the numerical lower bound), the experimental computations will focus on a different error estimator η based on an explicit gradient reconstruction. The orthogonal decomposition (3.5) implies that $\|R\|_b$ is the L^2 distance of ξ_h to Γ,

$$\|R\|_b = \min_{\gamma\in\Gamma} \|\xi_h - \gamma\|_b.$$

Thus, the explicit design of an approximation γ to ξ_h provides an upper bound. A direct approach for a reconstruction is to solve a discrete Laplacian so that $v_h \in P_{k+1}(\mathcal{T};\mathbb{R}^n)\cap V$ solves

$$(Dv_h, Dw_h)_b = (\xi_h, Dw_h)_b \quad \text{for all } w_h \in P_{k+1}(\mathcal{T};\mathbb{R}^n)\cap V.$$

With $\alpha_h := Dv_h$, the resulting computable error estimator reads

$$\eta := \|\alpha_h - \xi_h\|_b$$

and satisfies

$$\|R\|_b \leq \eta. \tag{5.2}$$

A standard efficiency analysis under smoothness assumptions shows that η can be expected to decrease at the same asymptotic rate as $\|R\|_b$. The design of α_h involves the solution of a global linear problem and should be understood as a proof of concept and not as the most efficient way of a gradient reconstruction.

5.2. Numerical experiments. The numerical experiments of this paper are concerned with rectangular two-dimensional domains. It was shown in [8] that on rectangular domains the nonzero part of the essential spectrum of the Cosserat operator equals

$$\{1\} \cup \left[\frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} + \frac{1}{\pi}\right], \tag{5.3}$$

and, consequently, there is a universal upper bound

$$\beta^2 \leq \frac{1}{2} - \frac{1}{\pi}$$

for all rectangular domains. It was analytically shown in [8] that isolated eigenfunctions exhibit strong corner singularities when the aspect ratio of the rectangle approaches 1. Thus, adaptive mesh refinement is used (as was done in [12]) in the numerical experiments. The refinement indicator is the residual-based error estimator η_{res} from (5.1), and the refinement is based on newest-vertex bisection with Dörfler marking [21] with bulk parameter $\theta = 0.3$. The polynomial degree is chosen as $k = 4$. Three aspect ratios are chosen, namely 2, 1.61, and 1. The tables display the number of elements in the triangulation, the computed eigenvalue μ_h, the eigenvalue error, and the empirical order of convergence (eoc), which compares the eigenvalue error with the number of triangles on two consecutive levels. The asymptotic lower bound (denoted by μ_h^{lb}) is computed based on $\lambda = 1/2$ in Assumption B, and reads, for $n = 2$ and the bound (5.2),

$$\mu_h^{\mathrm{lb}} = \mu_h - 2\sqrt{2\mu_h}\,\eta$$

and satisfies $\mu_h^{\mathrm{lb}} \leq \mu$ provided $b(\Pi\xi_h, \xi_h) \geq 1/2$ for the discrete b-normalized eigenfunction ξ_h. Furthermore, the error estimator η^2 as well as the ratio of η^2 and the eigenvalue error are shown.

TABLE 1
Results on the rectangle domain with aspect ratio 2.

card($\mathcal{T}$)	$10 \times \mu_h$	$\mu_h - \mu$	eoc	$10 \times \mu_h^{\mathrm{lb}}$	η^2	$\frac{\eta^2}{\mu_h - \mu}$
4	1.54455	4.48e-03	—	0.98980	2.49e-03	5.55e-01
8	1.53219	3.24e-03	4.65e-01	1.07879	1.67e-03	5.16e-01
12	1.52050	2.07e-03	1.10e+00	1.18206	9.41e-04	4.52e-01
20	1.51395	1.42e-03	4.28e-01	1.24489	5.97e-04	4.19e-01
39	1.51175	1.20e-03	5.68e-01	1.26869	4.88e-04	4.05e-01
64	1.50711	7.39e-04	9.82e-01	1.31912	2.93e-04	3.96e-01
70	1.50603	6.31e-04	9.20e-01	1.33477	2.43e-04	3.85e-01
100	1.50346	3.74e-04	1.90e+00	1.37074	1.46e-04	3.90e-01
112	1.50292	3.20e-04	1.37e+00	1.38169	1.22e-04	3.81e-01
136	1.50163	1.91e-04	2.64e+00	1.40692	7.46e-05	3.89e-01
148	1.50136	1.64e-04	1.81e+00	1.41466	6.25e-05	3.80e-01
172	1.50070	9.90e-05	3.37e+00	1.43250	3.87e-05	3.91e-01
192	1.50045	7.38e-05	2.66e+00	1.44187	2.85e-05	3.86e-01
226	1.50023	5.12e-05	2.24e+00	1.45104	2.01e-05	3.93e-01
246	1.50010	3.83e-05	3.42e+00	1.45773	1.49e-05	3.90e-01
284	1.49998	2.64e-05	2.56e+00	1.46459	1.04e-05	3.93e-01
304	1.49991	1.98e-05	4.24e+00	1.46938	7.76e-06	3.91e-01
328	1.49985	1.39e-05	4.67e+00	1.47385	5.63e-06	4.04e-01
348	1.49982	1.04e-05	4.78e+00	1.47721	4.25e-06	4.06e-01
372	1.49979	7.42e-06	5.17e+00	1.48031	3.16e-06	4.25e-01
396	1.49977	5.65e-06	4.36e+00	1.48262	2.45e-06	4.33e-01

5.3. Rectangle domain with aspect ratio 2. The rectangle $\Omega := (0,2)\times(0,1)$ with aspect ratio 2 is known to have a squared inf-sup constant $\mu = \beta^2$ that is well separated from the essential spectrum. There is an analytical upper bound due to [8, equation (5.5)] which reads $\mu \leq 0.166$. The reference value $\mu = 0.1499718$ was provided by [3], which together with (5.3) implies that Assumption A is satisfied with $\delta > 0.0317$. The velocity field belonging to the first eigenfunctions is known [8] to be not smoother than of class $H^{1.4760291}$, which together with the a priori error estimates from [12] implies an expected observed convergence rate of 0.47 for the error $\mu_h - \mu$ of eigenvalues under uniform mesh refinement, which was indeed observed in the computations of [12]. The computational results for adaptive meshes are displayed in Table 1. It was already observed in [12] that the mesh sequence is strongly refined near the corners of the rectangle and stays quite coarse in the interior (see [12] for a plot). Accordingly, the growth of the number of triangles card($\mathcal{T}$) with respect to the refinement level is rather slow. From the coarsest mesh on, the asymptotic lower bound μ_h^{lb} appears to be a true lower bound to μ. After 16 refinements, the adaptive algorithm shows the optimal rate of convergence with respect to the number of triangles. The ratio of the estimator η^2 and the eigenvalue error, however, is close to constant beginning from the first mesh.

5.4. Rectangle with aspect ratio 1.61. For the rectangle $\Omega := (0,1.61) \times (0,1)$ it is unknown whether the bottom of the Cosserat spectrum is an eigenvalue, but numerical evidence [12] indicates that there is indeed a spectral gap. The reference value from [12] reads $\mu = 0.18159009$ and implies an estimate $\delta > 10^{-4}$ for the spectral gap. The computational results are displayed in Table 2. The singularity of the eigenfunction is expected to be stronger than in the previous example, and, in this pre-asymptotic test, the optimal convergence rate is not attained within the first 19 refinements. As in the previous example, μ_h^{lb} converges to μ from below starting from the coarsest mesh, and the estimator η^2 appears to be reliable. In view of the small

TABLE 2
Results on the rectangle domain with aspect ratio 1.61.

card($\mathcal{T}$)	$10 \times \mu_h$	$\mu_h - \mu$	eoc	$10 \times \mu_h^{\mathrm{lb}}$	η^2	$\frac{\eta^2}{\mu_h - \mu}$
4	2.03623	2.20e-02	—	0.95309	7.20e-03	3.26e-01
8	1.99637	1.80e-02	2.87e-01	1.10623	4.96e-03	2.74e-01
12	1.96334	1.47e-02	4.98e-01	1.24683	3.26e-03	2.21e-01
31	1.94105	1.25e-02	1.72e-01	1.34014	2.32e-03	1.85e-01
54	1.91863	1.02e-02	3.55e-01	1.40864	1.69e-03	1.64e-01
78	1.89910	8.32e-03	5.73e-01	1.46206	1.25e-03	1.51e-01
102	1.88424	6.83e-03	7.33e-01	1.50502	9.54e-04	1.39e-01
126	1.87300	5.71e-03	8.50e-01	1.53899	7.44e-04	1.30e-01
150	1.86410	4.82e-03	9.71e-01	1.56739	5.90e-04	1.22e-01
174	1.85703	4.11e-03	1.06e+00	1.59109	4.76e-04	1.15e-01
198	1.85127	3.53e-03	1.16e+00	1.61126	3.88e-04	1.09e-01
222	1.84660	3.07e-03	1.23e+00	1.62813	3.23e-04	1.05e-01
246	1.84270	2.68e-03	1.32e+00	1.64285	2.70e-04	1.01e-01
270	1.83944	2.35e-03	1.39e+00	1.65569	2.29e-04	9.74e-02
294	1.83667	2.07e-03	1.47e+00	1.66699	1.95e-04	9.43e-02
318	1.83434	1.84e-03	1.51e+00	1.67674	1.69e-04	9.17e-02
342	1.83232	1.64e-03	1.59e+00	1.68544	1.47e-04	8.96e-02
366	1.83058	1.46e-03	1.65e+00	1.69323	1.28e-04	8.77e-02
390	1.82906	1.31e-03	1.72e+00	1.70022	1.13e-04	8.62e-02

value of δ in the order of 10^{-4}, this observation cannot be justified with Theorem 4.4, which provides a more pessimistic pre-asymptotic bound.

5.5. Square domain. For the square $\Omega := (0,1) \times (0,1)$, it is conjectured [9, 18] that $\mu = 1/2 - 1/\pi$, although the proof is still open. This experiment assumes that μ takes this value. In this case, Assumption A is not satisfied so that the numerical results are purely experimental and lack any justification by the proofs of this paper. The computational results are displayed in Table 3. The strong refinement close to one of the corners (see [12] for a plot) makes the number of triangles increase very slowly with respect to the levels, and, thus, not all refinement levels are shown. It can be seen that μ_h^{lb} converges monotonically from below starting from the coarsest mesh and that, remarkably, the error estimator η^2 shows reliability.

5.6. Conclusions from the computations. Beyond what is proven in this paper, the computational work gives insight into the performance of the numerical scheme in cases where the assumptions are questionable or at least not verifiable. The proofs of section 4 assume separation of $\beta^2 = \mu$ from the essential spectrum (Assumption A) and a sufficiently fine mesh-size so that $b(\Pi\xi_h, \xi_h)$ is larger than a certain constant. The pre-asymptotic experimental computations show that equivalence of the error with the error estimator is still observable when the gap δ is close or even equal to 0. Furthermore, the asymptotic lower eigenvalue bound is an actual lower bound in all computations.

Table 3
Results on the square domain.

card($\mathcal{T}$)	$10 \times \mu_h$	$\mu_h - \mu$	eoc	$10 \times \mu_h^{\mathrm{lb}}$	η^2	$\frac{\eta^2}{\mu_h - \mu}$
4	2.71586	8.98e-02	—	0.35988	2.55e-02	2.84e-01
12	2.56125	7.44e-02	1.71e-01	0.58033	1.91e-02	2.57e-01
24	2.36605	5.49e-02	4.38e-01	1.06010	9.01e-03	1.64e-01
44	2.23784	4.20e-02	4.38e-01	1.22345	5.74e-03	1.36e-01
52	2.16516	3.48e-02	1.13e+00	1.30232	4.29e-03	1.23e-01
60	2.10071	2.83e-02	1.43e+00	1.37300	3.15e-03	1.10e-01
68	2.05156	2.34e-02	1.51e+00	1.42838	2.36e-03	1.00e-01
76	2.01385	1.96e-02	1.57e+00	1.47236	1.81e-03	9.23e-02
84	1.98444	1.67e-02	1.61e+00	1.50801	1.42e-03	8.52e-02
92	1.96110	1.44e-02	1.64e+00	1.53745	1.14e-03	7.92e-02
100	1.94229	1.25e-02	1.67e+00	1.56212	9.30e-04	7.41e-02
⋮						
(25 refinement levels not displayed)						
⋮						
308	1.83337	1.64e-03	1.88e+00	1.74102	5.81e-05	3.52e-02
316	1.83259	1.56e-03	1.88e+00	1.74272	5.50e-05	3.50e-02
324	1.8318	1.49e-03	1.88e+00	1.74430	5.23e-05	3.49e-02
332	1.8311	1.42e-03	1.88e+00	1.74577	4.98e-05	3.48e-02
340	1.83056	1.36e-03	1.89e+00	1.74715	4.75e-05	3.48e-02
348	1.82998	1.30e-03	1.87e+00	1.74845	4.54e-05	3.46e-02
356	1.82943	1.25e-03	1.90e+00	1.74966	4.34e-05	3.47e-02

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