MIXED METHODS AND LOWER EIGENVALUE BOUNDS

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Abstract. It is shown how mixed finite element methods for symmetric positive definite eigenvalue problems related to partial differential operators can provide guaranteed lower eigenvalue bounds. The method is based on a classical compatibility condition (inclusion of kernels) of the mixed scheme and on local constants related to compact embeddings, which are often known explicitly. Applications include scalar second-order elliptic operators, linear elasticity, and the Steklov eigenvalue problem.

1. Introduction

Variationally posed symmetric eigenvalue problems related to positive definite partial differential equations (PDEs) with a compact resolvent are subject to the Rayleigh–Ritz principle [22], which characterizes the eigenvalues as certain minima in the corresponding Hilbert space. Consequently, conforming discretization methods, which are based on subspaces of the same Hilbert space, result in upper eigenvalue bounds. In contrast, the identification of guaranteed and efficient lower bounds to the eigenvalues is much more challenging, and general principles leading to lower bounds are not known. In the context of finite element methods (FEMs) for some PDE-based eigenvalue problems, major progress was achieved by [10, 19] where 'nonstandard' methods, i.e., methods beyond merely conforming discretizations, were employed to prove computable guaranteed lower bounds for the Laplacian. These approaches were later generalized to other eigenvalue problems in [8, 23, 15]. These methods have in common that they make use of the explicit knowledge of stability or approximation constants for projection operators related to the particular underlying finite element method.

By introducing a dual (or stress) variable, PDE eigenvalue problems can be posed in an equivalent mixed formulation; more precisely and following the terminology of [4], as a mixed problem of the second type. Stability properties as well as asymptotic error estimates for eigenvalue problems in mixed formulation are well understood, and the state of the art is documented in the review article [4] and the monograph [5]. Mixed formulations of positive definite problems are usually posed as saddle-point problems, which implies a higher computational cost compared with standard discretizations. While application-related advantages of operating with the mechanically relevant stress variable are sometimes mentioned for justifying and advertising mixed methods, a decisive structural advantage for systematically using mixed methods in eigenvalue computations has remained obscure. It is the aim of this contribution to reveal a basic and rather generic feature of dual mixed formulations and related finite element discretizations that allows for the computation of guaranteed lower eigenvalue bounds in many practical examples. The involved constants are related to properties of the underlying PDE operator rather than special properties of the discretization space. The applications

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presented in this paper include the Laplacian, general second-order scalar coercive operators, the Lamé eigenvalues of linear elasticity (where the present method seems to be the first in the literature to provide guaranteed lower eigenvalue bounds), and the Steklov eigenvalue problem. Generalizations to the Stokes or the biharmonic eigenvalue problem are possible and briefly outlined at the end of the paper.

The principal assumption on the discretization of the mixed system is the inclusion of kernels, a classical compatibility condition that for example guarantees that the equation for the divergence holds pointwise in the case of the Laplacian. In the usual mixed setting with discrete spaces Σ_h and $U_h \subseteq U$ and a bilinear form b, the inclusion of kernels implies that a projection P_h (in many cases the orthogonal projection) from U to U_h exists such that for any $v \in U$, $b(\cdot, v)$ and $b(\cdot, P_h v)$ represent the same linear functional over Σ_h. A consequence proven in this paper is the commutation property

$$\mathfrak{P}_h G = G_h P_h$$

where $\mathfrak{P}_h$ is the orthogonal projection to Σ_h with respect to a scalar product a and G is a gradient-like operator (from an integration by parts formula) with its discrete analogue G_h. In the simplest setting (without lower-order terms), the first discrete eigenvalue $\lambda_{1,h}$ is the minimum of the Rayleigh quotient $\|G_h v_h\|_a^2 / \|v_h\|_\ell^2$ for some seminorm $\|\cdot\|_\ell$ over appropriate elements v_h from U_h. Accordingly, the projection $P_h u$ of an ℓ-normalized first exact eigenfunction satisfies, as a candidate for the minimum,

$$\lambda_{1,h} \|P_h u\|_\ell^2 \leq \|G_h P_h u\|_a^2 \leq \lambda$$

because of the commutation property and $\|Gu\|_a^2 = \lambda$. Consequently, explicit control on the deviation of $\|P_h u\|_\ell^2$ from $\|u\|_\ell^2 = 1$ results in a guaranteed lower bound for λ. Under the assumption that there exists some δ_h such that $\|u - P_h u\|_\ell \leq \delta_h \|Gu\|_a$, the following guaranteed lower bound is established in Theorem 3.4

$$\frac{\lambda_h}{1 + \delta_h^2 \lambda_h} \leq \lambda.$$

In contrast to the methods proposed in [10, 19], this paper thereby provides a methodology that covers rather general operators in the sense that it basically requires the structure of saddle-point eigenvalue problems of the second type and some control on the generic projection P_h related to the inclusion-of-kernels property, but no particular knowledge on special interpolation or solution operators. It therefore covers a variety of eigenvalue problems with, e.g., variable coefficients and lower-order terms. A limitation of the approach as a post-processing method is that it intrinsically is a low-order method, which is also the case for the existing schemes [10, 19]. The reason is that the quantity δ_h^2 in the denominator of the lower bound is usually related to some compact embedding and does not improve when higher-order methods are employed. A direct computation shows that the difference of λ and the lower bound cannot be of a better order than $O(\delta^2)$. For the Laplacian, this limitation was recently overcome by [11], but the argument again uses particular properties of related finite element spaces and seems to be less universal.

This paper is organized as follows. Section 2 proves the fundamental commutation property for mixed methods with a compatibility condition (inclusion of kernels). The resulting abstract lower eigenvalue bound is shown in Section 3. The subsequent sections show applications to the Laplacian (Section 4), scalar elliptic operators (Section 5), linear elasticity (Section 6), and the Steklov eigenvalue problem (Section 7). Section 8 provides some conclusive remarks.

2. A basic commutation property

Let Σ, U be Hilbert spaces (the corresponding norms are denoted by $\|\cdot\|_\Sigma$ and $\|\cdot\|_U$) with bounded bilinear forms

$$a : \Sigma \times \Sigma \to \mathbb{R}, \quad b : \Sigma \times U \to \mathbb{R}.$$

Assume that a is symmetric and positive definite so that it induces a norm $\|\cdot\|_a = a(\cdot,\cdot)^{1/2}$, which is in general different from the norm in Σ. Let $\Sigma_h \subseteq \Sigma$, $U_h \subseteq U$ be finite-dimensional subspaces. We remark that the finite dimensional space Σ_h is also a Hilbert space when endowed with the inner product a. Given any $v \in U$, we write $G_h v \in \Sigma_h$ for the solution of the system

$$a(G_h v, \tau_h) = -b(\tau_h, v) \quad \text{for all } \tau_h \in \Sigma_h.$$

This system is uniquely solvable because a is an inner product on the finite-dimensional space Σ_h. In general, the analogue problem in the infinite-dimensional space Σ need not be solvable because Σ is not assumed complete and $b(\cdot, v)$ is not assumed continuous with respect to the norm $\|\cdot\|_a$. Let therefore $\overline{\Sigma}$ denote the closure of Σ with respect to $\|\cdot\|_a$ and let $U_0 \subseteq U$ denote the space of all elements v of U admitting a solution $Gv \in \overline{\Sigma}$ to

$$a(Gv, \tau) = -b(\tau, v) \quad \text{for all } \tau \in \Sigma. \tag{2.1}$$

The next lemma states that the space U_0 is a Hilbert space and satisfies an equivalence of norms provided the classical inf-sup condition [5] holds.

Lemma 2.1. *Let $(\cdot,\cdot)_U$ denote the inner product of U. The space U_0 with the inner product*

$$(v, w)_U + a(Gv, Gw) \quad \text{for any } v, w \in U_0$$

is a Hilbert space. If the inf-sup condition

$$0 < \beta = \inf_{v \in U\setminus\{0\}} \sup_{\tau \in \Sigma\setminus\{0\}} \frac{b(\tau, v)}{\|\tau\|_\Sigma \|v\|_U} \tag{2.2}$$

with some positive number $\beta > 0$ is satisfied, the norms $\|G\cdot\|_a$ and $(\|\cdot\|_U^2 + \|G\cdot\|_a^2)^{1/2}$ are equivalent on U_0.

Proof. For the proof that U_0 is a Hilbert space, it suffices to show that it is complete. Let $(u_j)_j$ be a sequence in U_0 such that $u_j \to u \in U$ with respect to $\|\cdot\|_U$ and $Gu_j \to \sigma \in \overline{\Sigma}$ with respect to $\|\cdot\|_a$, for $j \to \infty$. Then, for any $\tau \in \Sigma$,

$$a(\sigma, \tau) = a(\sigma - Gu_j, \tau) - b(\tau, u_j - u) - b(\tau, u).$$

The limit $j \to \infty$ and the continuity of a and b prove $a(\sigma, \tau) = -b(\tau, u)$ for any $\tau \in \Sigma$. Thus, $u \in U_0$ with $\sigma = Gu$.

For the proof of equivalence of norms, let C_a denote the continuity constant of a with respect to $\|\cdot\|_\Sigma$. The inf-sup condition (2.2) shows for any $v \in U_0$ that

$$\beta \|v\|_U \le \sup_{\tau \in \Sigma\setminus\{0\}} \frac{b(\tau, v)}{\|\tau\|_\Sigma} = \sup_{\tau \in \Sigma\setminus\{0\}} \frac{a(Gv, \tau)}{\|\tau\|_\Sigma} \le C_a^{1/2} \sup_{\tau \in \Sigma\setminus\{0\}} \frac{a(Gv, \tau)}{\|\tau\|_a} = C_a^{1/2} \|Gv\|_a.$$

This implies the asserted equivalence of norms. □

The following compatibility condition is essential to the subsequent arguments.

Condition 2.2. *The kernel*

$$Z := \{\tau \in \Sigma : b(\tau, v) = 0 \text{ for all } v \in U\}$$

and the discrete kernel

$$Z_h := \{\tau_h \in \Sigma_h : b(\tau_h, v_h) = 0 \text{ for all } v_h \in U_h\}$$

satisfy the inclusion $Z_h \subseteq Z$.

It is known [9, Lemma 2.3] that Condition 2.2 is equivalent to the existence of a projection $P_h : U \to U_h$ such that

$$b(\tau_h, v - P_h v) = 0 \quad \text{for all } (\tau_h, v) \in \Sigma_h \times U. \tag{2.3}$$

The argument is briefly repeated here for convenience of the reader.

Lemma 2.3. *Condition 2.2 is equivalent to the existence of a linear projection $P_h : U \to U_h$ satisfying* (2.3)*.*

Proof. The closed range theorem [6] in the finite-dimensional setting states

$$b(\cdot, U_h) = Z_h^0 \subseteq \Sigma_h^*,$$

namely that the functionals $b(\cdot, U_h)$ in the dual Σ_h^* of Σ_h represented by elements of U_h and the form b are precisely the elements of the polar set Z_h^0 of the kernel Z_h, i.e., the bounded linear functionals over Σ_h that vanish on Z_h. From the inclusion of kernels $Z_h \subseteq Z$ in Condition 2.2 we have that, given any $u \in U$, the functional $b(\cdot, u) \in \Sigma_h^*$ vanishes on Z_h and therefore belongs to Z_h^0, whence $b(\cdot, u) \in b(\cdot, U_h)$. Hence, there exists some $u_h \in U_h$ with $b(\cdot, u) = b(\cdot, u_h)$. This proves the existence of the projection $P_h u := u_h$. The element $P_h u$ can be chosen from the range of the discrete Riesz map $T_h : \Sigma_h \to U_h$ representing the from b via $b(\tau_h, \cdot) = (T_h \tau_h, \cdot)_U$ on U_h for any $\tau_h \in \Sigma_h$. With this choice, the projection P_h is linear. Conversely, it is immediate that the existence of the projection P_h implies Condition 2.2. □

Example 2.4. The example relevant to the applications in this paper is the following. Given $\tau \in \Sigma$, let $T\tau \in U$ denote the Riesz representation of $b(\tau, \cdot)$ in U such that any $v \in U$ satisfies $b(\tau, v) = (T\tau, v)_U$. Provided the inclusion $T\Sigma_h \subseteq U_h$ is satisfied, the projection $P_h : U \to U_h$ can be chosen as the orthogonal projection P_h to U_h with respect to the inner product of U.

Let $\mathfrak{P}_h : \Sigma \to \Sigma_h$ denote the orthogonal projection to the finite-dimensional space Σ_h with respect to the inner product a.

Lemma 2.5. *If Condition 2.2 holds, then any $v \in U_0$ satisfies $G_h P_h v = \mathfrak{P}_h G v$.*

Proof. Given $v \in U_0$, the definition of G_h and (2.3) show for any $\tau_h \in \Sigma_h$ that

$$a(G_h P_h v, \tau_h) = -b(\tau_h, P_h v) = -b(\tau_h, v) = a(Gv, \tau_h).$$

The last expression equals $a(\mathfrak{P}_h G v, \tau_h)$ and the lemma ensues. □

3. Approximation of the eigenvalue problem

We adopt the setting of Section 2 with the forms a and b, which are the ingredients of problems in the well-known saddle-point structure. It is assumed that b satisfies the inf-sup condition (2.2). It is additionally assumed that U_0 is dense in U. Let furthermore

$$c, \ell : U \times U \to \mathbb{R}$$

be two symmetric positive-semidefinite and bounded bilinear forms on U. In order to exclude the totally trivial case $\ell = 0$ (which would correspond to all eigenvalues equal to $+\infty$ in the system (3.1) below) it is assumed that there exists some $v_h \in U_h$ such that $\ell(v_h, v_h) > 0$ (which can be interpreted as a minimal resolution condition on the discrete space). The seminorms induced by c and ℓ are denoted by $\|\cdot\|_c = c(\cdot,\cdot)^{1/2}$ and $\|\cdot\|_\ell = \ell(\cdot,\cdot)^{1/2}$.

The eigenvalue problem seeks eigenpairs $(\lambda, u) \in \mathbb{R} \times U$ with nonzero u such that

$$a(\sigma, \tau) + b(\tau, u) = 0 \qquad \text{for all } \tau \in \Sigma \tag{3.1a}$$

$$b(\sigma, v) - c(u, v) = -\lambda\, \ell(u, v) \qquad \text{for all } v \in U. \tag{3.1b}$$

Note that the variable $\sigma = Gu$ is determined by u through (3.1a) and thus not treated as an independent variable. Recall Lemma 2.1, which states that U_0 is a Hilbert space. Since, for any $v \in U_0$, Gv and v satisfy $a(\sigma, Gv) = -b(\sigma, v)$ and the space U_0 is dense in U, the eigenvalue problem is equivalent to seeking eigenpairs $(\lambda, u) \in \mathbb{R} \times U_0$ satisfying, for all $v \in U_0$,

$$\mathcal{A}(u,v) = \lambda\, \ell(u,v) \quad \text{where} \quad \mathcal{A}(u,v) := a(Gu, Gv) + c(u,v).$$

The left-hand side defines an inner product on U_0 and, hence, the eigenfunctions u corresponding to finite eigenvalues are $\mathcal{A}$-orthogonal to the kernel of ℓ and will henceforth be normalized as $\|u\|_\ell = 1$. We assume that the solution operator mapping $f \in U$ to the solution $(\sigma, u) \in \Sigma \times U$ to the linear problem

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= 0 && \text{for all } \tau \in \Sigma \\ b(\sigma,v) - c(u,v) &= -\ell(f,v) && \text{for all } v \in U \end{aligned}$$

is a compact operator (assuming the solution being measured in the norm $(\|u\|_U^2 + \|\sigma\|_a^2)^{1/2}$). Therefore, the finite part of the spectrum consists of eigenvalues that have no finite accumulation point and can be enumerated $0 < \lambda_1 \le \lambda_2 \le \dots$. There exists an orthonormal set of corresponding eigenfunctions, which will henceforth be referred to as "the eigenfunctions". The Rayleigh quotient for the smallest eigenvalue reads

$$\lambda_1 = \min_{\substack{v \in U_0 \setminus \{0\} \\ v \perp_{\mathcal{A}} \ker \ell}} \frac{\|Gv\|_a^2 + \|v\|_c^2}{\|v\|_\ell^2}.$$

Here, $\perp_{\mathcal{A}}$ denotes orthogonality with respect to $\mathcal{A}$ and $\ker \ell$ is the space of all $v \in U$ with $\ell(v,v) = 0$. The higher eigenvalues satisfy analogous min-max principles, the discrete version of which is displayed as (3.3) below.

For the choice of discrete spaces, we assume that b satisfies a discrete inf-sup condition

$$0 < \beta_h = \inf_{v_h \in U_h \setminus \{0\}} \sup_{\tau_h \in \Sigma_h \setminus \{0\}} \frac{b(\tau_h, v_h)}{\|\tau_h\|_\Sigma \|v_h\|_U}$$

with respect to the finite-dimensional spaces Σ_h and U_h. The discrete eigenvalue problem seeks discrete eigenpairs $(\lambda_h, u_h) \in \mathbb{R} \times U_h$ with $\|u_h\|_\ell = 1$ such that

$$a(\sigma_h, \tau_h) + b(\tau_h, u_h) = 0 \quad \text{for all } \tau_h \in \Sigma_h \tag{3.2a}$$

$$b(\sigma_h, v_h) - c(u_h, v_h) = -\lambda_h\, \ell(u_h, v_h) \quad \text{for all } v_h \in U_h. \tag{3.2b}$$

Analogous arguments as above show that the discrete eigenvalue problem is equivalent to

$$\mathcal{A}_h(u_h, v_h) = \lambda_h\, \ell(u_h, v_h) \quad \text{where} \quad \mathcal{A}_h(u_h, v_h) := a(G_h u_h, G_h v_h) + c(u_h, v_h).$$

The bilinear form $\mathcal{A}_h$ on the left-hand side defines an inner product on U_h. The finite discrete eigenvalues are enumerated as $0 < \lambda_{1,h} \le \lambda_{2,h} \le \dots \le \lambda_{N,h}$ for some positive integer N. The first discrete eigenvalue minimizes the following Rayleigh quotient

$$\lambda_{1,h} = \min_{\substack{v_h \in U_h \setminus \{0\} \\ v_h \perp_{\mathcal{A}_h} \ker \ell}} \frac{\|G_h v_h\|_a^2 + \|v_h\|_c^2}{\|v_h\|_\ell^2}$$

where $\perp_{\mathcal{A}_h}$ denotes orthogonality with respect to $\mathcal{A}_h$. More generally, the Jth discrete eigenvalue satisfies the min-max principle [22]

$$\lambda_{J,h} = \min_{\substack{V_J \subseteq U_h \\ \dim(V_J) = J, V_J \perp_{\mathcal{A}_h} \ker \ell}} \max_{v_h \in V_J \setminus \{0\}} \frac{\|G_h v_h\|_a^2 + \|v_h\|_c^2}{\|v_h\|_\ell^2} \tag{3.3}$$

where the minimum is taken over all J-dimensional subspaces of U_h that are $\mathcal{A}_h$-orthogonal to the kernel of ℓ. Sufficient conditions on the spaces Σ_h and U_h such that the discrete eigenvalues $\lambda_{j,h}$ approximate the true eigenvalues λ_j are well known [4]. The focus of this work is the computation of guaranteed lower bounds to the eigenvalues λ_j for an index $j \in \{1, \ldots, J\}$.

The key condition required for the theory in this paper is the following.

Condition 3.1. *There exists a number* $\delta_h = \delta_h(\Sigma_h, U_h)$ *such that any element* $u \in \operatorname{span}\{u_1, \ldots, u_J\}$ *in the linear hull of the first* J *eigenfunctions satisfies*

$$\|u - P_h u\|_\ell^2 \leq \delta_h^2 (\|Gu\|_a^2 + \|u\|_c^2).$$

It is furthermore required that the projection P_h is compatible with c and ℓ in the following sense.

Condition 3.2. *Any element* $u \in \operatorname{span}\{u_1, \ldots, u_J\}$ *in the linear hull of the first* J *eigenfunctions satisfies the Pythagorean identity*

$$\text{(i)} \qquad \|P_h u\|_\ell^2 + \|u - P_h u\|_\ell^2 = \|u\|_\ell^2,$$

the stability estimate

$$\text{(ii)} \qquad \|P_h u\|_c^2 \leq \|u\|_c^2,$$

and the orthogonality

$$\text{(iii)} \qquad P_h u \perp_{\mathcal{A}_h} \ker \ell.$$

Remark 3.3. In all practical examples listed in this paper the requirements from Condition 3.2 can be verified for any $u \in U$.

Theorem 3.4 (abstract lower bound)**.** *Let* Σ, U *be Hilbert spaces with a symmetric and positive definite bilinear form* a *on* Σ *and a continuous bilinear form* $b : \Sigma \times U \to \mathbb{R}$ *such that the space* $U_0 \subseteq U$ *of admissible right-hand sides for* (2.1) *is dense. Let the inf-sup condition* (2.2)*, the inclusion of kernels from Condition 2.2 as well as Condition 3.1 and Condition 3.2 be satisfied. Let* $\Sigma_h \subseteq \Sigma$ *and* $U_h \subseteq U$ *be an inf-sup stable pair of finite-dimensional subspaces and let* c *and* ℓ *be continuous bilinear forms on* U *such that* ℓ *acts nontrivially on* U_h *and there exist at least* J *finite discrete eigenvalues* $\lambda_{1,h}, \ldots, \lambda_{J,h}$ *to* (3.2) *for a positive integer* J*. Let* λ_J *denote the* J*th eigenvalue to* (3.1) *with an* ℓ*-normalized eigenfunction* u_J*. Then, the following lower bound for* λ_J *holds*

$$\frac{\lambda_{J,h}}{1 + \delta_h^2 \lambda_{J,h}} \leq \lambda_J.$$

Proof. Let $u_1, \ldots, u_J$ denote a basis of ℓ-normalized eigenfunctions corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_J$. In a first step, we assume that the projected eigenfunctions $(P_h u_j : j = 1, \ldots, J)$ are linear independent. Since the projected eigenfunctions are linear independent and, by Condition 3.2(iii) they are $\mathcal{A}_h$-orthogonal to the kernel of ℓ, they form an admissible J-dimensional subspace $\tilde{V}_J$ and there are coefficients $\alpha_1, \ldots, \alpha_J$, normalized to $\sum_j \alpha_j^2 = 1$, such that

$$v_h := P_h u \quad \text{for } u := \sum_{j=1}^{J} \alpha_j u_j$$

maximizes the Rayleigh quotient over $\tilde{V}_J$. The discrete Rayleigh quotient (3.3), elementary properties of the minimum, Lemma 2.5, the nonexpansivity of orthogonal projections, and Condition 3.2(ii) imply

$$\|v_h\|_\ell^2 \lambda_{J,h} \leq \|G_h v_h\|_a^2 + \|v_h\|_c^2 \leq \Big\| \sum_{j=1}^{J} \alpha_j G u_j \Big\|_a^2 + \Big\| \sum_{j=1}^{J} \alpha_j u_j \Big\|_c^2 = \lambda_J$$

because different eigenfunctions are mutually $\mathcal{A}$-orthogonal. Condition 3.2, Condition 3.1, and the normalization of the coefficients α_j show

$$\|v_h\|_\ell^2 = \|u\|_\ell^2 - \|u - P_h u\|_\ell^2 \geq 1 - \delta_h^2 \lambda_J.$$

The combination of the two foregoing displayed formulas results in

$$(1 - \delta_h^2 \lambda_J)\lambda_{J,h} \leq \lambda_J.$$

Rearranging this formula yields the asserted lower eigenvalue bound.

In the remaining case that the projected eigenfunctions $(P_h u_j : j = 1, \dots, J)$ are linear dependent, we apply an idea from [21]. There exist coefficients $\beta_1, \dots, \beta_J$ with $\sum_{j=1}^J \beta_j^2 = 1$ and a linear combination $u := \sum_{j=1}^J \beta_j u_j$ with $\|u\|_\ell = 1$ and $P_h u = 0$. Condition 3.1 implies

$$1 = \|u\|_\ell^2 = \|u - P_h u\|_\ell^2 \leq \delta_h^2 (\|Gu\|_a^2 + \|u\|_c^2) \leq \delta_h^2 \lambda_J.$$

The last inequality follows from the $\mathcal{A}$-orthogonality of the eigenfunctions and the normalization of the coefficients β_j. Since, in particular, $\delta_h > 0$, we infer $1/\delta_h^2 \leq \lambda_J$. Elementary estimates lead to

$$\frac{\lambda_{J,h}}{1 + \delta_h^2 \lambda_{J,h}} \leq \frac{\lambda_{J,h}}{\delta_h^2 \lambda_{J,h}} = \frac{1}{\delta_h^2} \leq \lambda_J,$$

which implies the asserted lower eigenvalue bound. □

4. Application to the Laplacian

This section is to fix notation and to present the application of Theorem 3.4 to the eigenvalues of the Laplacian. Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded, connected, polytopal Lipschitz domain. The eigenvalue problem for the Dirichlet-Laplacian seeks eigenpairs (λ, u) with

$$-\Delta u = \lambda u \quad \text{in } \Omega \quad u = 0 \text{ on } \partial\Omega$$

where the eigenfunction $u \in H_0^1(\Omega)\setminus\{0\}$ belongs to the first-order L^2-based Sobolev space with vanishing trace on the boundary, and the Laplacian Δ is understood in the sense of weak derivatives. The mixed formulation is based on the choice

$$\Sigma = H(\operatorname{div}, \Omega) \quad \text{and} \quad U = L^2(\Omega)$$

where $L^2(\Omega)$ is the space of square-integrable measurable functions and $H(\operatorname{div}, \Omega)$ is the space of vector fields over Ω whose components as well as distributional divergence belong to $L^2(\Omega)$. The inner product in (any power of) $L^2(\Omega)$ is denoted by $(\cdot,\cdot)_{L^2(\Omega)}$. The form a is defined as the L^2 inner product of vector fields, $a(\cdot,\cdot) := (\cdot,\cdot)_{L^2(\Omega)}$, and b is defined by

$$b(\tau, v) := (\operatorname{div} \tau, v)_{L^2(\Omega)} \quad \text{for any } (\tau, v) \in \Sigma \times U.$$

With the choice $c = 0$ and $\ell(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}$, it is well known [4, 5] that the eigenvalues of the Laplacian correspond to those of system (3.1) and that the form b satisfies the inf-sup condition (2.2).

Let Σ_h and U_h be an inf-sup stable pair of finite-dimensional spaces with the property $\operatorname{div} \Sigma_h \subseteq U_h$, which guarantees Condition 2.2. It is assumed that the discrete spaces are related to a partition $\mathcal{T}$ of $\bar{\Omega}$ in convex polytopes (for example a simplicial triangulation) and that the space $P_0(\mathcal{T})$ of piecewise constant functions over $\mathcal{T}$ is contained in U_h. The most prominent example of such a pair is the choice of Σ_h as the lowest-order Raviart–Thomas space with respect to $\mathcal{T}$ and $U_h = P_0(\mathcal{T})$, but many other choices are possible [5]. Since the piecewise constants are contained in U_h, we have

$$\|u - P_h u\|_U \leq \|u - \Pi_{0,h} u\|_{L^2(\Omega)}$$

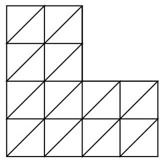

FIGURE 1. Initial triangulation of the L-shaped domain.

$h \times \sqrt{2}$	$\lambda_{1,h}$	lower bound	upper bound
2^0	8.60144	5.99088	13.1991
2^{-1}	9.25186	8.28147	10.5739
2^{-2}	9.49208	9.21512	9.91654
2^{-3}	9.58268	9.51054	9.72837
2^{-4}	9.61746	9.59919	9.66981

TABLE 1. Results for the Laplace eigenvalues on the L-shaped domain.

for the L^2 projection $\Pi_{0,h}$ onto the piecewise constants. Each element $T \in \mathcal{T}$ of the partition is convex, whence the constant of the Poincaré inequality is explicitly known [20] and equals h_T/π with the diameter $h_T := \operatorname{diam}(T)$ of T. Therefore

$$\|u - P_h u\|_U \leq \frac{h}{\pi}\|\nabla u\|_{L^2(\Omega)} = \frac{h}{\pi}\|Gu\|_{L^2(\Omega)}$$

for the maximal element diameter $h := \max_{T\in\mathcal{T}} h_T$, where it has been used that $u \in U_0$ possesses a weak gradient in $[L^2(\Omega)]^n$. This verifies Condition 3.1 with $\delta_h = h/\pi$. Condition 3.2 is trivially satisfied because ℓ is the L^2 inner product and $c = 0$. In conclusion, Theorem 3.4 applies and the resulting lower bound for the Laplacian reads as follows.

Corollary 4.1 (guaranteed lower eigenvalue bound for the Laplacian)**.** *Assume the above setting for the mixed formulation of the Dirichlet-Laplacian. Let $\Sigma_h \subseteq \Sigma$, $U_h \subseteq U$ be an inf-sup stable pair of finite-dimensional subspaces related to a partition $\mathcal{T}$ in convex polytopes with $\mathcal{P}_0(\mathcal{T}) \subseteq U_h$ and $\operatorname{div}\Sigma_h \subseteq U_h$. Then, the Jth eigenvalue λ_J of* (3.1) *and the Jth discrete eigenvalue $\lambda_{J,h}$ of* (3.2) *satisfy*

$$\frac{\lambda_{J,h}}{1 + (h^2/\pi^2)\lambda_{J,h}} \leq \lambda_J.$$

Remark 4.2. It is known that for $n = 2$ and triangular partitions the constant of the Poincaré inequality can be slightly improved [18]. In this case, π^2 in Corollary 4.1 can be replaced by $j_{1,1}^2$ where $j_{1,1}$ is the first root of the Bessel function of the first kind.

Example 4.3. Consider the first eigenvalue of the Dirichlet-Laplacian on the L-shaped domain $(-1,1)^2 \setminus [0,1]^2$. Let $\mathcal{T}$ be a triangulation of Ω and let Σ_h be the lowest-order Raviart–Thomas finite element space [5], which is the subspace of $H(\operatorname{div},\Omega)$ of vector fields that, when restricted to any $T \in \mathcal{T}$, are linear combinations of constants and the identity $x \mapsto x$. The corresponding space $U_h = P_0(\mathcal{T})$ is the space of piecewise constants. The initial triangulation is displayed in Figure 1. Table 1 displays the discrete eigenvalue, the guaranteed lower bound from Corollary 4.1, and an upper bound computed with a first-order conforming FEM on a sequence of uniformly refined meshes. The computed bound is that from Corollary 4.1 and disregards the slight improvement mentioned in Remark 4.2.

5. Scalar elliptic operator

As a generalization of the eigenvalue problem from the previous section we consider the eigenvalue problem

$$-\operatorname{div}(A\nabla u) + \gamma u = \lambda u \quad \text{in } \Omega \quad u = 0 \text{ on } \partial\Omega. \tag{5.1}$$

Here A is a symmetric matrix field over Ω with $L^\infty(\Omega)$ coefficients satisfying the bounds

$$a_0|\xi|^2 \leq \xi^T A\xi \leq a_1|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n \quad \text{a.e. in } \Omega$$

with real numbers $0 < a_0 \leq a_1 < \infty$; and $\gamma \in L^\infty(\Omega)$ is a nonnegative function with $0 \leq \gamma_0 \leq \gamma \leq \gamma_1$ almost everywhere. As in the Laplacian case, the spaces for the mixed formulation are

$$\Sigma = H(\operatorname{div}, \Omega) \quad \text{and} \quad U = L^2(\Omega).$$

For simplicity it is assumed that A and γ are piecewise constant with respect to a given triangulation $\mathcal{T}$, which will also be used for the discretization.

The mixed formulation is based on the substitution $\sigma = A\nabla u$. The form a is defined as $a(\cdot,\cdot) := (\cdot, A^{-1}\cdot)_{L^2(\Omega)}$, and b is defined by

$$b(\tau, v) := (\operatorname{div}\tau, v)_{L^2(\Omega)} \quad \text{for any } (\tau, v) \in \Sigma \times U.$$

With the choice $c(\cdot,\cdot) = (\cdot, \gamma\cdot)_{L^2(\Omega)}$ and $\ell(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}$, it is not difficult to verify that system (3.1) is then inf-sup stable and equivalent to the original problem (5.1).

Let Σ_h and U_h be an inf-sup stable pair of finite-dimensional spaces with the property $\operatorname{div}\Sigma_h \subseteq U_h$. It is again assumed that the discrete spaces are related to a partition $\mathcal{T}$ of $\bar{\Omega}$ in convex polytopes and that U_h contains the piecewise constant functions $P_0(\mathcal{T})$. In addition, it is assumed that U_h does not include a constraint on inter-element continuity. More precisely, it is assumed that U_h is of the structure

$$U_h = \prod_{T\in\mathcal{T}} V_T \tag{5.2}$$

where V_T is a subspace of $L^2(T)$ and the embedding $L^2(T) \subseteq L^2(\Omega)$ is understood through extensions by zero. This assumption ensures that the L^2 projection to U_h localizes to the elements of $\mathcal{T}$. This property is used for the verification of the stability property from Condition 3.2(ii): Since the L^2 projection onto U_h equals the local L^2 projection, we have

$$\|P_h u\|_c^2 \leq \sum_{T\in\mathcal{T}} \gamma|_T \, \|\Pi_{h,T} u\|_{L^2(T)}^2 \leq \sum_{T\in\mathcal{T}} \gamma|_T \|u\|_{L^2(T)}^2 = \|u\|_c^2.$$

For verifying Condition 3.1 and determining the constant, we use the local Poincaré inequality and infer

$$\|u - P_h u\|_\ell^2 = \|u - \Pi_{0,h} u\|_{L^2(\Omega)}^2 \leq \frac{h^2}{\pi^2}\|\nabla u\|_{L^2(\Omega)}^2 \leq \frac{h^2}{a_0\pi^2}\|A^{1/2}\nabla u\|_{L^2(\Omega)}^2$$

so that Condition 3.1 is satisfied with $\delta_h = h/(a_0^{1/2}\pi)$. Theorem 3.4 therefore implies the following result.

Corollary 5.1 (guaranteed lower eigenvalue bound for elliptic operators)**.** *Assume the above setting for the mixed formulation of* (5.1)*. Let $\Sigma_h \subseteq \Sigma$, $U_h \subseteq U$ be an inf-sup stable pair of finite-dimensional subspaces related to a partition $\mathcal{T}$ in convex polytopes with $\mathcal{P}_0(\mathcal{T}) \subseteq U_h$ and $\operatorname{div}\Sigma_h \subseteq U_h$ where U_h is a space without interelement continuity requirements in the sense of the structure from* (5.2)*. Then, the Jth eigenvalue λ_J of* (3.1) *and the Jth discrete eigenvalue $\lambda_{J,h}$ of* (3.2) *satisfy*

$$\frac{\lambda_{J,h}}{1 + \lambda_{J,h} h^2/(a_0\pi^2)} \leq \lambda_J.$$

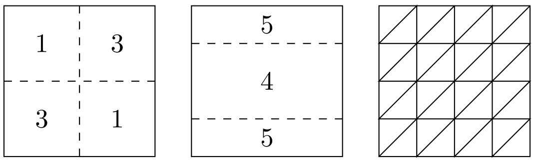

FIGURE 2. Coefficients A (left) and γ (middle) and the initial triangulation (right) in Example 5.3.

$h \times \sqrt{2}$	$\lambda_{1,h}$	lower bound	upper bound
2^{0}	13.4656	10.3977	15.4049
2^{-1}	13.4010	12.4008	13.9124
2^{-2}	13.3898	13.1187	13.5205
2^{-3}	13.3877	13.3185	13.4207
2^{-4}	13.3873	13.3699	13.3956

TABLE 2. Results for eigenvalue problem (5.1) of the scalar elliptic operator on the square domain.

In the case that the lower bound γ_0 to the low-order coefficient γ in the elliptic eigenvalue problem is positive, one can take advantage of a spectral shift and obtain a sharper lower bound.

Corollary 5.2 (guaranteed lower eigenvalue bound for elliptic operators with shift)**.** *Under the assumptions of Corollary 5.1, the Jth eigenvalue λ_J of* (3.1) *and the Jth discrete eigenvalue $\lambda_{J,h}$ of* (3.2) *satisfy*

$$\frac{\lambda_{J,h}}{1+(\lambda_{J,h}-\gamma_0)h^2/(a_0\pi^2)}+\gamma_0\frac{(\lambda_{J,h}-\gamma_0)h^2/(a_0\pi^2)}{1+(\lambda_{J,h}-\gamma_0)h^2/(a_0\pi^2)}\leq\lambda_J.$$

Proof. The eigenvalues $\hat\lambda$ of the shifted problem

$$-\operatorname{div}(A\nabla u)+(\gamma-\gamma_0)u=\hat\lambda u\quad\text{in }\Omega\quad u=0\text{ on }\partial\Omega$$

are related to the ones of (5.1) by $\hat\lambda_j+\gamma_0=\lambda_j$ and an analogous shift property applies to the discrete problem so that $\hat\lambda_{j,h}+\gamma_0=\lambda_{j,h}$. Since $\gamma-\gamma_0$ is nonnegative, Corollary 5.1 applies and proves

$$\frac{\hat\lambda_{J,h}}{1+\hat\lambda_{J,h}h^2/(a_0\pi^2)}\leq\hat\lambda_J.$$

Equivalently,

$$\frac{\lambda_{J,h}-\gamma_0}{1+(\lambda_{J,h}-\gamma_0)h^2/(a_0\pi^2)}+\gamma_0\leq\lambda_J.$$

This implies the asserted lower bound. □

Example 5.3. On the square domain $\Omega=(-1,1)^2$ choose the coefficients

$$A(x)=\left(2+\frac{x_1x_2}{|x_1|\,|x_2|}\right)I_{2\times2}\quad\text{and}\quad\gamma(x)=4+1_{\{|x_2|>1/2\}}$$

where $I_{2\times2}$ is the two-dimensional unit matrix. The lower bounds on the coefficients read $a_0=1$ and $\gamma_0=4$. The coefficients and the initial triangulation are displayed in Figure 2. Table 2 compares the discrete eigenvalues, the guaranteed lower bound from Corollary 5.2, and upper bounds from a conforming standard FEM on a sequence of uniformly refined meshes.

6. Application to linear elasticity

Let Ω be a domain as in the previous sections with a disjoint partition $\partial\Omega = \Gamma_D \cup \Gamma_N$ of the boundary where Γ_D is assumed, for simplicity of the presentation, to have positive surface measure. For the sake of a simple exposition, it is furthermore assumed that the parts Γ_D and Γ_N are resolved by the boundary faces of some underlying polytopal partition $\mathcal{T}$ of $\bar{\Omega}$. The linear elasticity eigenvalue problem seeks eigenvalues λ and vector-valued eigenfunctions $u \neq 0$ such that

$$-\operatorname{div} \mathbb{C}\varepsilon(u) = \lambda u \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \Gamma_D \qquad \mathbb{C}\varepsilon(u)\mathbf{n} = 0 \quad \text{on } \Gamma_N.$$

Here, $\mathbf{n}$ is the outward pointing unit vector to Γ_N, $\varepsilon(u) = \frac{1}{2}(Du + (Du)^T)$ is the symmetric part of the derivative matrix, and the elasticity tensor $\mathbb{C}$ reads

$$\mathbb{C}(A) = 2\mu A + \kappa \operatorname{tr}(A) I_{n\times n} \quad \text{for any symmetric matrix } A$$

for given material parameters $\mu, \kappa > 0$ and the n-dimensional unit matrix $I_{n\times n}$. The action of the divergence to a $n \times n$ matrix field is understood row-wise. The mixed formulation is based on the space $\Sigma := H_{\Gamma_N}(\operatorname{div}, \Omega; \mathbb{S})$ of symmetric matrix fields σ whose rows belong to $H(\operatorname{div}, \Omega)$ and that satisfy the homogeneous Neumann boundary condition $\sigma\mathbf{n} = 0$ on Γ_N; and $U := [L^2(\Omega)]^n$. The bilinear forms a, b are defined as $a(\cdot,\cdot) := (\cdot, \mathbb{C}^{-1}\cdot)_{L^2(\Omega)}$, and

$$b(\tau, v) := (\operatorname{div} \tau, v) \quad \text{for any } (\tau, v) \in \Sigma \times U$$

while $c = 0$ and $\ell(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}$. It is known that this is an inf-sup stable formulation of the linear elastic eigenvalue problem [3, 2, 5].

Let Σ_h and U_h be an inf-sup stable pair of finite-dimensional spaces with $\operatorname{div} \Sigma_h \subseteq U_h$. Instances of such spaces with pointwise symmetry for the stress field can be based on piecewise polynomials [3, 2] or on piecewise rational trial functions [16]. It is assumed that the discrete spaces are related to a partition $\mathcal{T}$ of $\bar{\Omega}$ in convex polytopes and that the space

$$\Big\{ v \in [L^2(\Omega)]^n : \text{for any } T \in \mathcal{T}, D(v|_T) \text{ is constant and skew-symmetric} \Big\}$$

of piecewise infinitesimal rigid body motions with respect to $\mathcal{T}$ is contained in U_h. Since the infinitesimal rigid-body motions on an element $\mathcal{T}$ include all constants, the Poincaré inequality yields on any $T \in \mathcal{T}$ for the L^2 projection $\Pi_{\mathrm{RM},h}$ onto the piecewise infinitesimal rigid-body motions,

$$\|u - \Pi_{\mathrm{RM},h} u\|_{L^2(T)} \leq \frac{h_T}{\pi} \|D(u - \Pi_{\mathrm{RM},h} u)\|_{L^2(T)}.$$

Korn's inequality on T with constant $C_K(T)$ then yields

$$\|u - \Pi_{\mathrm{RM},h} u\|_{L^2(T)} \leq \frac{h_T}{\pi} \|D(u - \Pi_{\mathrm{RM},h} u)\|_{L^2(T)} \leq \frac{C_K(T) h_T}{\pi} \|\varepsilon(u)\|_{L^2(T)}.$$

Thus, with $1/(2\mu)$ as the smallest eigenvalue of the elasticity tensor $\mathbb{C}$ and

$$\delta_h := \frac{\max_{T\in\mathcal{T}} C_K(T) h_T}{\sqrt{2\mu}\pi}$$

it follows that

$$\|u - P_h u\|_U^2 \leq \frac{\max_{T\in\mathcal{T}} C_K(T)^2 h_T^2}{\pi^2} \|\varepsilon(u)\|_{L^2(\Omega)}^2 \leq \delta_h^2 \|\mathbb{C}^{1/2}\varepsilon(u)\|_{L^2(\Omega)}^2$$

which verifies Condition 3.1.

Corollary 6.1 (guaranteed lower eigenvalue bound for elasticity)**.** *Assume the above setting for the mixed formulation of the elasticity system. Let $\Sigma_h \subseteq \Sigma$, $U_h \subseteq U$ be an inf-sup stable pair of finite-dimensional subspaces related to a partition $\mathcal{T}$ in convex polytopes with* $\operatorname{div} \Sigma_h \subseteq U_h$ *where U_h contains the piecewise infinitesimal* *rigid-body motions. Then, the Jth eigenvalue λ_J of* (3.1) *and the Jth discrete eigenvalue $\lambda_{J,h}$ of* (3.2) *satisfy*

$$\frac{\lambda_{J,h}}{1 + (\max_{T\in\mathcal{T}} C_K(T)^2 h_T^2)/(2\mu\pi^2))\lambda_{J,h}} \le \lambda_J.$$

For practical and guaranteed bounds, upper bounds on the local Korn constants $C_K(T)$ are needed. In two dimensions, upper bounds for $C_K(T)$ on convex polygons can be explicitly computed from the bound on the continuity constant of a right-inverse of the divergence operator available in [13, Section 5.1.2]. It is known that the latter has a close relation to the Korn constant, and the precise argument is given as follows. Let $\omega \subseteq \mathbb{R}^n$ be an open, bounded, connected Lipschitz domain. It is well known [1] that there exists a constant $C_{\mathrm{div}} < \infty$ such that any $p \in L^2(\Omega)$ with $\int_\omega p\,dx = 0$ can be represented as $p = \operatorname{div} v$ with a vector field $v \in [H_0^1(\Omega)]^n$ with $\|Dv\|_{L^2(\Omega)} \le C_{\mathrm{div}}\|p\|_{L^2(\Omega)}$. If $\omega \subseteq \mathbb{R}^2$ is in addition a convex planar polygon with corners $z_1, \dots, z_m$, a fixed (arbitrary) interior point $x_0 \in \omega$, and the geometric parameter

$$d := \frac{\operatorname{dist}(x_0, \partial\omega)}{\max_{j=1,\dots,m} |x_0 - z_j|}, \tag{6.1}$$

then the following bound provided by [13] is valid

$$C_{\mathrm{div}} \le \sqrt{\frac{2}{d^2}(1 + \sqrt{1-d^2})}. \tag{6.2}$$

In what follows, we will always choose x_0 as the centre of the largest incribed ball of the polyhedron. Then, for example, any right-isosceles triangle $\hat{T}$ satisfies $d(\hat{T}) = 1/\sqrt{4 + 2\sqrt{2}}$ and, accordingly,

$$C_{\mathrm{div}}(\hat{T}) \le 5.1259.$$

The following lemma shows how the bound on C_{div} can be used for bounding the Korn constant in two space dimensions. The usual rotation of two-dimensional vector fields reads $\operatorname{rot} v = \partial_2 v_1 - \partial_1 v_2$.

Lemma 6.2 (explicit bound on local Korn inequality in 2D)**.** *Let $\omega \subseteq \mathbb{R}^2$ be a bounded, open, convex polygon with the geometric parameter d from* (6.1) *and let $v \in [H^1(\omega)]^2$ be a vector field with $\int_\omega \operatorname{rot} v\,dx = 0$. Then*

$$\|Dv\|_{L^2(\omega)} \le \sqrt{1 + \frac{4}{d^2}(1 + \sqrt{1-d^2})}\|\varepsilon(v)\|_{L^2(\omega)}.$$

Proof. The tensor field $\tau = Dv$ is irrotational, which is equivalent to the fact that the field $\tau^\perp := (-\tau_{12}, \tau_{11}; -\tau_{22}, \tau_{21})$ is divergence-free. Further, the property $\int_\omega \operatorname{rot} v\,dx = 0$ is equivalent to $\int_\omega \operatorname{tr}\tau^\perp\,dx = 0$. By a classical argument [5, Proposition 9.1.1] it can be shown that

$$\|\operatorname{tr}\tau^\perp\|_{L^2(\Omega)} \le 2C_{\mathrm{div}}\|\tau^\perp - \frac{1}{2}\operatorname{tr}\tau^\perp I\|_{L^2(\Omega)}.$$

This implies, with $I_\perp := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ that

$$\|\tau_{21} - \tau_{12}\|_{L^2(\Omega)} \le 2C_{\mathrm{div}}\|\tau - \frac{1}{2}(\tau_{21} - \tau_{12})I_\perp\|_{L^2(\Omega)} = 2C_{\mathrm{div}}\|\frac{1}{2}(\tau + \tau^T)\|_{L^2(\Omega)}$$

so that the skew-symmetric part of τ is controlled by the symmetric part of τ. From the orthogonality of symmetric and skew-symmetric matrices we then infer with $\varepsilon(v) = \frac{1}{2}(\tau + \tau^T)$ that

$$\|\tau\|^2_{L^2(\Omega)} = \|\varepsilon(v)\|^2_{L^2(\Omega)} + \|\frac{1}{2}(\tau_{21} - \tau_{12})I_\perp\|^2_{L^2(\Omega)} \le (1 + 2C^2_{\mathrm{div}})\|\varepsilon(v)\|^2_{L^2(\Omega)}.$$

The asserted estimate then follows with the bound (6.2) on C_{div}. $\square$

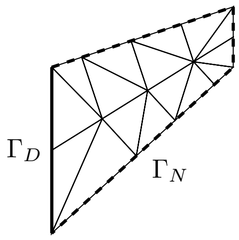

FIGURE 3. Cook's membrane with initial trianulation.

h	$\lambda_{1,h}$	lower bound	upper bound
33.14	5.62812e-4	4.40411e-5	1.00153e-3
16.57	5.68410e-4	1.43028e-4	9.07262e-4
8.287	5.71724e-4	3.27099e-4	7.68356e-4
4.143	5.73587e-4	4.82990e-4	6.58116e-4
2.071	5.74507e-4	5.48734e-4	6.05333e-4

TABLE 3. Results for the elasticity eigenvalues on Cook's membrane.

Lemma 6.2 shows that the Korn constant on a convex polygon ω can be bounded as

$$C_K \leq \sqrt{1 + \frac{4}{d^2}(1 + \sqrt{1 - d^2})}.$$

For example, given a right-isosceles triangle $\hat{T}$, the value of the Korn constant satisfies the bound

$$C_K(\hat{T}) \leq 7.318.$$

Remark 6.3. For domains with sufficiently regular boundary, there exists a sharper alternative to the the bound of Lemma 6.2, see [17] and the references therein.

Example 6.4. The Dirichlet boundary Γ_D of Cook's membrane $\Omega \subseteq \mathbb{R}^2$ is given by the straight line from $(0, 0)$ to $(0, 44)$. The domain Ω is given as the interior of the convex combination of Γ_D with the points $(48, 44)$ and $(48, 60)$, and, accordingly, $\Gamma_N = \partial\Omega \setminus \Gamma_D$ is the Neumann boundary. The domain with its initial triangulation is displayed in Figure 3. In the numerical example, the Lamé parameters are chosen as $\mu = 1$ and $\kappa = 100$. The spaces Σ_h and U_h are taken according to the Arnold–Winther finite element method [3] based on a regular triangulation $\mathcal{T}$, that is, Σ_h is the subspace of Σ consisting of symmetric matrix fields whose components, when restricted to any triangle of T, are at most cubic polynomials on T, and whose divergence is piecewise affine, while U_h is the space of piecewise affine vector fields. Table 3 displays the discrete eigenvalue, the guaranteed lower bound from Corollary 4.1, and an upper bound computed with a first-order conforming FEM on a sequence of uniformly refined meshes. It should be remarked, that the optimal order of convergence of the Arnold–Winther method is better than linear, so that in general the guaranteed lower bound including the global mesh size h is expected to be sub-optimal if sufficient smoothness of the eigenfunctions is available. This effect is not visible in this experiment because of the Dirichlet–Neumann corners in the configuration of the boundary, which lead to reduced regularity. There exist lower-order methods respecting the symmetry of stresses (see, e.g., [3, 16]), but their implementation is not necessarily easier compared with the usual Arnold–Winther finite element.

Remark 6.5. A similar reasoning yields lower eigenvalue bounds for the Stokes system, which corresponds to the formal limit $\kappa \to \infty$. It is known that the mixed formulation of the Lamé system is robust (locking-free) with respect to this limit.

Remark 6.6. The technique from [13] for bounding C_{div} extends to domains ω that are star-shaped with respect to all points of some open nonempty ball $B \subseteq \omega$.

Remark 6.7. The stated bounds on Korn's constant do not apply to three-dimensional element domains. Upper bounds can be numerically computed with the method of [14], but their theoretical justification relies on (asymptotic) assumptions the verification of which turns out difficult in practice.

7. Application to the Steklov eigenvalue problem

Let $\Omega \subseteq \mathbb{R}^n$ be a domain as in prior sections with outer unit normal $\mathbf{n}$. As a model problem we consider the problem of finding (λ, w) with nontrivial w such that

$$\begin{aligned} -\Delta w + w &= 0 \qquad && \text{in } \Omega \\ \frac{\partial w}{\partial \mathbf{n}} &= \lambda w && \text{on } \partial\Omega. \end{aligned}$$

The eigenvalue relation on the boundary subject to a homogeneous linear partial differential equation in the domain is related to the spectrum of a Dirichlet-to-Neumann map. The standard variational formulation is posed in the Sobolev space $H^1(\Omega)$. Since no dual mixed formulation has been studied in the literature so far, it is explained here in more detail than the classical models of the foregoing sections. The idea is to introduce the variables $\sigma = \nabla w$ and (w, γ) with $\gamma = w|_{\partial\Omega}$. Let $\Sigma := H_\Gamma(\operatorname{div}, \Omega)$ be the subspace of all $\tau \in H(\operatorname{div}, \Omega)$ whose normal trace $\tau \cdot \mathbf{n}|_{\partial\Omega}$ belongs to $L^2(\partial\Omega)$, equipped with the norm

$$\|\tau\|_{H_\Gamma(\operatorname{div},\Omega)} := \sqrt{\|\tau\|^2_{H(\operatorname{div},\Omega)} + \|\tau \cdot \mathbf{n}\|^2_{L^2(\partial\Omega)}}$$

and let $U := L^2(\Omega) \times L^2(\partial\Omega)$. Let $a(\cdot,\cdot) := (\cdot,\cdot)_{L^2(\Omega)}$ be chosen as the L^2 product and let

$$b(\tau, (v, \eta)) := (\operatorname{div} \tau, v)_{L^2(\Omega)} - (\tau \cdot \mathbf{n}, \eta)_{L^2(\partial\Omega)}.$$

With $c(\cdot,\cdot) := (\cdot,\cdot)_{L^2(\Omega)}$ and $\ell(\cdot,\cdot) := (\cdot,\cdot)_{L^2(\partial\Omega)}$, the Steklov eigenvalue problem can then be rewritten as system (3.1). For convenience, the eigenvalue problem is explicitly rewritten in the following: Seek $(\sigma, (w, \gamma)) \in \Sigma \times U$ and $\lambda \in \mathbb{R}$ such that

$$\begin{array}{lclcll} (\sigma, \tau)_{L^2(\Omega)} & + & (\operatorname{div} \tau, w)_{L^2(\Omega)} & - & (\tau \cdot \mathbf{n}, \gamma)_{L^2(\partial\Omega)} & = 0 \\ (\operatorname{div} \sigma, v)_{L^2(\Omega)} & - & (w, v)_{L^2(\Omega)} & & & = 0 \\ -(\sigma \cdot \mathbf{n}, \eta)_{L^2(\partial\Omega)} & & & & & = -\lambda(\gamma, \eta)_{L^2(\partial\Omega)} \end{array}$$

for all $(\tau, (v, \eta)) \in \Sigma \times U$. It is not difficult to see that the system is inf-sup stable and the finite eigenvalues coincide with those of the original system. Furthermore, the relations $\operatorname{div} \sigma = w$ and $\sigma \cdot \mathbf{n} = \lambda\gamma$ allow for substitutions in the first row of the system resulting in the equivalent eigenvalue problem

$$(\sigma, \tau)_{L^2(\Omega)} + (\operatorname{div} \sigma, \operatorname{div} \tau)_{L^2(\Omega)} = \lambda^{-1}(\sigma \cdot \mathbf{n}, \tau \cdot \mathbf{n})_{L^2(\Omega)} \quad \text{for all } \tau \in \Sigma.$$

An analogous equivalence can be used for the discretization and results in a positive definite system matrix, which is beneficial from a practical point of view.

The model discretization presented here is based on a partition $\mathcal{T}$ in convex polytopes, a subspace $\Sigma_h \subseteq \Sigma$ from an inf-sup stable pair (Σ_h, V_h) for the Laplacian (as in Section 4) with the compatibility condition $\operatorname{div}\Sigma_h \subseteq V_h$, and $U_h := V_h \times \operatorname{tr}_{\partial\Omega}\Sigma_h$ (the symbol $\operatorname{tr}_{\partial\Omega}$ refers to the normal trace). This implies the relation

$$(\operatorname{div}, \operatorname{tr}_{\partial\Omega})\Sigma_h \subseteq U_h$$

sufficient for Condition 2.2 to hold. The verification of Condition 3.2 is immediate because the orthogonal projection in U_h is the product of the orthogonal projections with respect to the components w, γ. The assumption that the trace variable is discretized with $\operatorname{tr}_{\partial\Omega}\Sigma_h$ allows for a reduction to a positive-definite system as outlined above for the continuous setting.

Example 7.1. On simplicial triangulations, the simplest example is the pairing of the lowest-order Raviart–Thomas space Σ_h and the product space of piecewise constants with respect to $\mathcal{T}$ and the piecewise constants with respect to the boundary faces $\mathcal{F}(\partial\Omega)$, written

$$U_h := P_0(\mathcal{T}) \times P_0(\mathcal{F}(\partial\Omega)).$$

The inf-sup stability of the discrete system and the compatibility condition are then consequences of standard results on the Raviart–Thomas element [5].

In what follows it is assumed that U_h contains the subspace $P_0(\mathcal{T}) \times P_0(\mathcal{F}(\partial\Omega))$ of piecewise constant functions. In order to verify Condition 3.1, it then suffices to determine a constant δ_h such that

$$\|w - \Pi_{0,\mathcal{F}(\partial\Omega)} w\|^2_{L^2(\partial\Omega)} \le \delta_h^2 (\|\nabla w\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)})$$

where $\Pi_{0,\mathcal{F}(\partial\Omega)}$ is the L^2 projection onto the piecewise constants with respect to the boundary faces. Note that $w \in H^1(\Omega)$ so that $w|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$. This is achieved with the following trace inequality for convex polytopes in terms of the geometry of an inscribed simplex.

Lemma 7.2. *Let $\omega \subseteq \mathbb{R}^n$ be a convex polytope with a face F and let $T \subseteq \omega$ be an inscribed simplex with F as one of its faces. Let $v \in H^1(\omega)$ with $\int_F v\,ds = 0$. Then*

$$\|v\|_{L^2(F)} \le \sqrt{\frac{\operatorname{meas}_{n-1}(F)}{\operatorname{meas}_n(T)}}\, h_T \sqrt{\frac{n+2\pi}{n\pi^2}}\, \|\nabla v\|_{L^2(\omega)}$$

where the symbol $\operatorname{meas}_{n-1}(F)$ *denotes the* $(n-1)$*-dimensional surface measure of* F *and* $\operatorname{meas}_n(T)$ *denotes the volume of* T*.*

Proof. Without loss of generality one may assume $\int_T v\,dx = 0$ because $\|v\|_{L^2(F)} \le \|v-c\|_{L^2(F)}$ for any real constant c, in particular for the volume average $c = ⨍_T v\,dx$. Let P denote the vertex of T opposite to F. The integration-by-parts formula with the outer unit normal $\mathbf{n}_T$ to ∂T implies

$$n\int_T v^2\,dx + \int_T (\bullet - P)\cdot\nabla(v^2)\,dx = \int_{\partial T} v^2(\bullet - P)\cdot\mathbf{n}_T\,ds.$$

The vector $(x-P)$ is tangential to ∂T for almost all $x \in \partial T \setminus F$ and thus orthogonal to $\mathbf{n}_T$. If x belongs to the interior of F, an elementary geometric consideration (volume of a cone in $\mathbb{R}^n$) shows that $(x-P)\cdot\mathbf{n}_T = n\operatorname{meas}_n(T)/\operatorname{meas}_{n-1}(F)$. Therefore the integral on the right-hand side equals $n\operatorname{meas}_n(T)/\operatorname{meas}_{n-1}(F)\int_F v^2\,dx$. This leads to the classical trace identity

$$\begin{aligned}\int_F v^2\,ds &= \frac{\operatorname{meas}_{n-1}(F)}{\operatorname{meas}_n(T)}\int_T v^2\,dx + \frac{\operatorname{meas}_{n-1}(F)}{n\operatorname{meas}_n(T)}\int_T (\bullet - P)\cdot\nabla(v^2)\,dx\\ &\le \frac{\operatorname{meas}_{n-1}(F)}{\operatorname{meas}_n(T)}\int_T v^2\,dx + \frac{2h_T\operatorname{meas}_{n-1}(F)}{n\operatorname{meas}_n(T)}\int_T |v|\,|\nabla v|\,dx.\end{aligned}$$

Using the Young inequality with an arbitrary scaling parameter $\alpha > 0$, the second term on the right-hand side can be bounded as follows

$$\frac{2h_T \operatorname{meas}_{n-1}(F)}{n \operatorname{meas}_n(T)} \int_T |v|\,|\nabla v|\,dx \leq \frac{h_T \operatorname{meas}_{n-1}(F)}{n \operatorname{meas}_n(T)} \left(\alpha \|v\|^2_{L^2(T)} + \frac{1}{\alpha} \|\nabla v\|^2_{L^2(T)} \right).$$

Together with the Poincaré bound $\|v\|_{L^2(T)} \leq (h_T/\pi)\|\nabla v\|_{L^2(T)}$ this yields

$$\|v\|^2_{L^2(F)} \leq \frac{\operatorname{meas}_{n-1}(F)}{\operatorname{meas}_n(T)} \left(\frac{h_T^2(1 + h_T\alpha/n)}{\pi^2} + \frac{h_T}{n} \frac{1}{\alpha} \right) \|\nabla v\|^2_{L^2(T)}.$$

For $\alpha := \pi/h_T$ we obtain

$$\|v\|^2_{L^2(F)} \leq \frac{\operatorname{meas}_{n-1}(F)}{\operatorname{meas}_n(T)} h_T^2 \left(\frac{1}{\pi^2} + \frac{2}{\pi n} \right) \|\nabla v\|^2_{L^2(T)}.$$

□

If m is the maximal possible number of faces of a polytope K of $\mathcal{T}$ and $\mathcal{T}$ is not a singleton set, an overlap argument and Lemma 7.2 show that Condition 3.1 is satisfied with

$$\delta_h = \sqrt{m-1} \max_{\substack{K \in \mathcal{T} \\ F \subseteq T \subseteq K}} \sqrt{\frac{\operatorname{meas}_{n-1}(F)}{\operatorname{meas}_n(T)}} h_T \sqrt{\frac{n + 2\pi}{n\pi^2}}.$$

The notation $F \subseteq T \subseteq K$ indicates that F is a boundary face and there exists a simplex T inscribed to the polytope $K \in \mathcal{T}$ such that F is simultaneously a face of T and K.

Corollary 7.3 (guaranteed lower Steklov eigenvalue bound)**.** *Assume the above setting for the mixed formulation of the Steklov eigenproblem. Let $\Sigma_h \subseteq \Sigma$, $U_h \subseteq U$ be an inf-sup stable pair of finite-dimensional subspaces related to a partition $\mathcal{T}$ (with at least two elements) in convex polytopes (with at most m faces) with* $(\operatorname{div}, \operatorname{tr}_{\partial\Omega})\Sigma_h \subseteq U_h$ *where U_h contains the piecewise constants $P_0(\mathcal{T}) \times P_0(\mathcal{F}(\partial\Omega))$. Then, the Jth eigenvalue λ_J of* (3.1) *and the Jth discrete eigenvalue $\lambda_{J,h}$ of* (3.2) *satisfy*

$$\frac{\lambda_{J,h}}{1 + (m-1) \max\limits_{\substack{K \in \mathcal{T} \\ F \subseteq T \subseteq K}} \frac{\operatorname{meas}_{n-1}(F)}{\operatorname{meas}_n(T)} h_T^2 \frac{n + 2\pi}{n\pi^2} \lambda_{J,h}} \leq \lambda_J.$$

Remark 7.4. Under shape-regularity assumptions, in the bound of Corollary 7.3 the prefactor of $\lambda_{h,J}$ in the denominator is proportional to h.

Example 7.5. In this example, the Steklov eigenvalue problem in two dimensions is discretized with the lowest-order Raviart–Thomas finite element pairing from Example 7.1 with respect to a regular triangulation. The domain under consideration is the L-shaped domain from Example 4.3 with the initial triangulation from Figure 1. Table 4 displays the discrete eigenvalue, the guaranteed lower bound from Corollary 7.3, and an upper bound computed with a first-order conforming FEM, on a sequence of uniformly refined meshes.

8. Conclusive remarks

There are many more applications of Theorem 3.4 beyond the model problems highlighted in this paper. For example, the Stokes eigenvalues in the formulation without symmetry constraint on the stress can be discretized in a dual pseudostress formulation [7], which can be applied for the computation of guaranteed lower eigenvalue bounds of the Stokes system, with or without lower order terms. Another example is the biharmonic eigenvalue problem, where a dual mixed method has

$h \times \sqrt{2}$	$\lambda_{1,h}$	lower bound	upper bound
2^0	0.340304	0.188241	0.344375
2^{-1}	0.341129	0.242816	0.342217
2^{-2}	0.341342	0.283844	0.341624
2^{-3}	0.341397	0.309994	0.341469
2^{-4}	0.341411	0.324951	0.341430

TABLE 4. Results for the Steklov eigenvalues on the L-shaped domain.

been provided by [12]. A computable quantity δ_h then requires knowledge on the fundamental frequency of the biharmonic operator with free boundary condition over reference polyhedra. Those can be numerically computed as in [8].

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