THE TRANSPOSITION METHOD IN SADDLE-POINT PROBLEMS

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Abstract. The transposition method known from the theory of second-order elliptic partial differential equations is formulated for saddle-point problems under certain structural assumptions known from the Brezzi splitting plus mapping properties that reflect elliptic regularity in concrete instances of differential operators with boundary conditions. A framework is devised that grants approximation by standard, unmodified Galerkin-type methods, which may be nonconforming with respect to the more general data allowed by the transposed operator. That the data need not be regularized is an advantage particular to the saddle-point approach. Examples covered by the framework include the Laplacian, the Stokes operator, linear elasticity, the bi-Laplacian, and the stationary Maxwell problem with L^2 boundary data.

1. Introduction

The transposition method is an elementary mechanism that converts *regularity of solutions* to an elliptic partial differential equation (PDE) for certain classes of right-hand sides to the *regularity of test functions* of the adjoint (transposed) operator. It was described in [32] and [29, 27] for second-order elliptic operators with boundary data in L^2, which do not admit solutions with bounded Dirichlet energy. The numerical approximation by finite element methods (FEM) was studied by [9, 2, 3, 22, 25]. The use of primal methods requires a fit of the boundary data to the trace space because boundary conditions are imposed in an essential fashion. This is not necessary when mixed dual methods are used because the Dirichlet datum appears on the right-hand side only. The compatibility issue is then shifted to the fact that the data may not be compatible with the mapping properties of the trace. As an effect, the discretized saddle-point problem may then be well-posed without further regularization: a decisive advantage of the mixed method. Its convergence as an approximation scheme, however, deserves justification. In particular, the mixed method, though not modified compared to its original form, may become nonconforming when applied to the transposed problem because the discrete functions may not form a subspace of the smoother test functions of the transposed problem. As an illustrating example, we mention Poisson's problem with L^2 boundary data, where the very weak solution is defined by testing with functions from the space $H^{1+s}(\Omega)$ for some $s > 1/2$. Their gradients then belong to H^s, but the usual discrete Raviart–Thomas vector fields only belong to H^ν with $\nu < 1/2$. In [25], the mixed Raviart–Thomas method was justified for the Laplacian boundary value problem with L^2 data, where the convergence analysis used the regularization framework of [2]. In this work we give an alternative and to

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some extent more structure-oriented approach that circumvents any regularization in the method and its error analysis.

The arguments used here are fairly generic in the sense that they are structural on the level of functional analysis of saddle-point problems, in spite of elliptic PDE being the principal application. We therefore propose a framework that departs from a stable saddle-point problem in the sense of Brezzi [11] in Hilbert spaces $\Sigma \times U$, where Σ is a subspace of a possibly larger Hilbert space H with $\Sigma \subseteq H$. Given data $g \in \Sigma'$ and $f \in U'$, the saddle-point problem seeks $(\sigma, u) \in \Sigma \times U$ such that

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= \langle g,\tau\rangle \\ b(\sigma,v) &= -\langle f,v\rangle \end{aligned} \qquad \text{for all } (\tau,v) \in \Sigma \times U.$$

Assuming standard conditions on a, b from the Brezzi splitting, this problem is well posed, and boundary data of u in standard PDE applications will describe subclasses of Σ'. The well known idea of transposition [32] is to use elliptic regularity and a transposed isomorphism, which results in an equation with more regular test functions, thereby giving a meaning to less regular data. Since the mixed formulation of elliptic problems shifts the boundary data from an essential condition to a datum on the right-hand side, we can make use of the fact that the discrete equations are well-defined whenever the discrete right-hand side is. In this work we aim at an error analysis of the resulting approximations under generic assumptions on mixed discretizations, which is Theorem 1. It states an error estimate for the approximation of u in the norm of U, which in the PDE context this will often be the L^2 norm. We give a series of instances in particular PDEs to illustrate the implication to actual methods.

The remaining parts of this article are organized as follows. Section 2 presents, as our main result, an error estimate for the mixed Galerkin-type methods for the problem with irregular data g. Applications to specific model problems are then detailed in the subsequent sections, including the Laplacian (Section 3), the Stokes problem (Section 4), linear elasticity (Section 5), the biharmonic equation (Section 6), and the Maxwell equation (Section 7).

Throughout this work, an inequality $A \le CB$ with a constant C that is independent of discretization parameters is denoted by $A \lesssim B$.

2. Framework, discretization, main result

This section develops the transposition method for saddle-point problems and analyzes a class of possibly nonconforming mixed discretizations together with error estimates.

2.1. The continuous mixed problem.

We start with a basic structural setting of saddle-point problems.

Assumption 1 (structure)**.** (i) Let Σ, U be Hilbert spaces (the corresponding norms are denoted by $\|\cdot\|_\Sigma$ and $\|\cdot\|_U$) with bounded bilinear forms

$$a : \Sigma \times \Sigma \to \mathbb{R}, \quad b : \Sigma \times U \to \mathbb{R}.$$

Let H be a Hilbert space with the norm $\|\cdot\|_H$ such that $\Sigma \hookrightarrow H$ continuously. We assume that a extends to a bounded, symmetric, positive semidefinite form on $H \times H$. Thus, $|a(\sigma,\tau)| \lesssim \|\sigma\|_H \|\tau\|_H$ for all $\sigma, \tau \in H$.

(ii) The Brezzi conditions hold: inf-sup of b and Σ-coercivity of a on the kernel of b, that is, there exist constants $\alpha_0 > 0$ and $\beta_0 > 0$ such that

$$\inf_{0\neq\tau\in Z} \|\tau\|_\Sigma^{-2} a(\tau,\tau) \ge \alpha_0 \quad \text{and} \quad \inf_{0\neq v\in U} \sup_{0\neq\tau\in\Sigma} (\|\tau\|_\Sigma \|v\|_U)^{-1} b(\tau,v) \ge \beta_0,$$

where $Z := \{\tau \in \Sigma : b(\tau, v) = 0 \text{ for all } v \in U\}$ denotes the kernel of b. □

The coercivity of a on the kernel of b, together with the inf-sup condition for b are sufficient for the well-posedness of the mixed problem [11].

We identify U' with U through the Riesz isomorphism of U in this paper. Let $\langle \cdot, \cdot \rangle$ denote the duality pairing of $\Sigma' \times \Sigma$. For every $g \in \Sigma'$ and $f \in U'$, consider the mixed problem: find $(\sigma, u) \in \Sigma \times U$ such that

$$\begin{cases} a(\sigma, \tau) + b(\tau, u) = \langle g, \tau \rangle & \text{for all } \tau \in \Sigma, \\ b(\sigma, v) = -(f, v)_U & \text{for all } v \in U. \end{cases} \tag{2.1}$$

Assumption 1 and the saddle-point theory [11] imply that (2.1) has a unique solution and

$$\|\sigma\|_\Sigma + \|u\|_U \lesssim \|g\|_{\Sigma'} + \|f\|_U. \tag{2.2}$$

Accordingly, we define the continuous saddle-point operator

$$T : \Sigma \times U \to \Sigma' \times U$$

by $T(\sigma, u) = (g, f)$ if and only if (σ, u) satisfies (2.1). By Assumption 1 and (2.2), T is an isomorphism.

2.2. **The transposition method for the mixed formulation.** We are interested in irregular boundary data $g \notin \Sigma'$. In this setting, the original mixed formulation is not defined and the transposition method for the mixed formulation is invoked.

Assumption 2 (elliptic regularity)**.** Assume we are given Banach subspaces $\Sigma^r \subseteq \Sigma$ and $U^r \subseteq U$ endowed with stronger norms $\|\cdot\|_{\Sigma^r}$ and $\|\cdot\|_{U^r}$, respectively. For given $z \in U$, any solution to $T(\phi, w) = (0, z)$ satisfies

$$\|\phi\|_{\Sigma^r} + \|w\|_{U^r} \lesssim \|z\|_U. \tag{2.3}$$

□

By Assumption 2, we can define the transposition solution as follows.

Definition 2.1. Let Assumptions 1–2 hold. Let $f \in U$ and let $g \in (\Sigma^r)'$. An element $u \in U$ is called the transposition solution associated with data (f, g), if

$$(u, z)_U = (f, w)_U - \langle g, \phi \rangle \qquad \text{for all } z \in U, \tag{2.4}$$

where (ϕ, w) is determined by $T(\phi, w) = (0, z)$. For any $z \in U$, the extended boundary value of the transposition solution $u \in U$ is characterized by

$$\langle \gamma u, \phi \rangle := (f, w)_U - (u, z)_U,$$

for every pair (ϕ, w) satisfying $T(\phi, w) = (0, z)$.

Lemma 2.2 (transposition solution)**.** *Assume that Assumptions 1 and 2 hold. For every $f \in U$ and $g \in (\Sigma^r)'$, there exists a unique transposition solution $u \in U$ to (2.4). If $g \in \Sigma'$, then the transposition solution coincides with the solution of the classical mixed formulation $T(\sigma, u) = (g, f)$ for some $\sigma \in \Sigma$.*

Proof. Given $z \in U$, Assumption 2 implies that there exists a unique solution $(\phi, w) \in \Sigma^r \times U^r$ to $T(\phi, w) = (0, z)$. Define the linear functional $\mathcal{F}(z) := (f, w)_U - \langle g, \phi \rangle$ for $z \in U$. The regularity estimate (2.3) and $U^r \hookrightarrow U$ continuously show

$$|\mathcal{F}(z)| \le \|g\|_{(\Sigma^r)'} \|\phi\|_{\Sigma^r} + \|f\|_U \|w\|_U \lesssim \big(\|g\|_{(\Sigma^r)'} + \|f\|_U\big) \|z\|_U.$$

Thus $\mathcal{F}$ is continuous on U. By the Riesz isomorphism, there exists a unique $u \in U$ such that $(u, z)_U = \mathcal{F}(z)$ for all $z \in U$. Moreover,

$$\|u\|_U = \|\mathcal{F}\|_{U'} \lesssim \|f\|_U + \|g\|_{(\Sigma^r)'}.$$

This proves the existence, uniqueness and stability of the transposition solution. It remains to show that for $g \in \Sigma'$ this reproduces solutions to the original problem. For $g \in \Sigma'$, Assumption 1 shows that there exists a unique $(\sigma, u_c) \in \Sigma \times U$ satisfying $T(\sigma, u_c) = (g, f)$. Taking $\tau = \phi$ in the first equation of $T(\sigma, u_c) = (g, f)$ gives $\langle g, \phi\rangle = a(\sigma, \phi) + b(\phi, u_c)$. Choosing $\tau = \sigma$ in the first equation of $T(\phi, w) = (0, z)$ and using the symmetry of $a(\cdot, \cdot)$ yields $a(\sigma, \phi) + b(\sigma, w) = 0$. The second equation of $T(\sigma, u_c) = (g, f)$ then implies $(f, w)_U = a(\sigma, \phi)$. Consequently,

$$(f, w)_U - \langle g, \phi\rangle = a(\sigma, \phi) - \big(a(\sigma, \phi) + b(\phi, u_c)\big) = -b(\phi, u_c).$$

Using the second equation of $T(\phi, w) = (0, z)$ with $v = u_c$ gives $b(\phi, u_c) = -(z, u_c)_U$. Hence,

$$(f, w)_U - \langle g, \phi\rangle = (z, u_c)_U = (u_c, z)_U \quad \text{for all } z \in U.$$

Therefore, u_c satisfies the defining identity of the transposition solution. The uniqueness of the transposition solution completes the proof. □

Remark 2.3. The transposition solution from Lemma 2.2 satisfies $\langle \gamma u, \phi\rangle = \langle g, \phi\rangle$ only for solutions to $T(\phi, w) = (0, z)$ with $z \in U$. The relation $\gamma u = g$ as an identity in $(\Sigma^r)'$ would require the additional condition that T is an isomorphism between $\Sigma^r \times U^r$ and $\{0\} \times U$.

2.3. **Mixed discretization.** Let families of finite-dimensional spaces $\Sigma_h \subset H$ and $U_h \subset U$, where h is a formal parameter that can be thought of as a discretization length-scale in practice. Let $\|\cdot\|_{\Sigma,h}$ be a seminorm on $\Sigma_h + \Sigma$, which coincides with $\|\cdot\|_\Sigma$ for elements of Σ and which is a norm on Σ_h. We do not assume $\Sigma_h \subset \Sigma$ here, but assume that the embedding $\Sigma_h \hookrightarrow H$ is uniform in h. Assume that we are given a bilinear form b_h on $(\Sigma_h + \Sigma) \times U$ with $b_h(\tau, \cdot) = b(\tau, \cdot)$ for all $\tau \in \Sigma$ and $|b_h(\tau_h, v)| \lesssim \|\tau_h\|_{\Sigma,h}\|v\|_U$ for all $\tau_h \in \Sigma_h$.

We recall that both Σ^r and Σ_h are subspaces of H, and so is their sum. Abusing notation, given a linear functional on $\Sigma_h + \Sigma^r$, we write $g \in \Sigma_h' \cap (\Sigma^r)'$ if $\|g\|_{\Sigma_h'}$ and $\|g\|_{(\Sigma^r)'}$ are finite, the norms being the usual operator norms with respect to $\|\cdot\|_{\Sigma,h}$ and $\|\cdot\|_{\Sigma^r}$, respectively. The space of such g is denoted by $\Sigma_h' \cap (\Sigma^r)'$.

The discrete mixed problem reads: Given $g \in \Sigma_h' \cap (\Sigma^r)'$ and $f \in U$, find $(\sigma_h, u_h) \in \Sigma_h \times U_h$ such that

$$\begin{cases} a(\sigma_h, \tau_h) + b_h(\tau_h, u_h) = \langle g, \tau_h\rangle & \text{for all } \tau_h \in \Sigma_h, \\ b_h(\sigma_h, v_h) = -(f, v_h)_U & \text{for all } v_h \in U_h. \end{cases} \tag{2.5}$$

Assumption 3 (discrete stability)**.** The discrete formulation (2.5) is well-posed: There exist constants $\alpha_1 > 0$ and $\beta_1 > 0$, independent of h, such that

$$\inf_{0\neq\tau_h\in Z_h} \|\tau_h\|_{\Sigma,h}^{-2} a(\tau_h, \tau_h) \geq \alpha_1, \quad \inf_{0\neq v_h\in U_h} \sup_{0\neq\tau_h\in\Sigma_h} (\|\tau_h\|_{\Sigma,h}\|v_h\|_U)^{-1} b_h(\tau_h, v_h) \geq \beta_1$$

with $Z_h := \{\tau_h \in \Sigma_h : b_h(\tau_h, v_h) = 0 \text{ for all } v_h \in U_h\}$.

Assumption 3 as a standard condition in saddle-point theory [11] implies that (2.5) has a unique solution and that

$$\|\sigma_h\|_{\Sigma,h} + \|u_h\|_U \lesssim \|g\|_{\Sigma_h'} + \|f\|_U. \tag{2.6}$$

In what follows, three further quantitative assumptions on the discretization spaces are given for the sake of proving error estimates of the discrete solution (σ_h, u_h) as an approximation to the transposition solution u.

Assumption 4 (uniform approximation of U^r)**.** Assume there exists a linear continuous operator $P_h : U^r \to U_h$ and a function $\rho_U = \rho_U(h)$ with $\rho_U \to 0$ as $h \to 0$ satisfying

$$\|w - P_h w\|_U \lesssim \rho_U \|w\|_{U^r} \quad \text{for all } w \in U^r.$$

Assumption 5 (commuting approximation of Σ^r)**.** Assume there exists a linear and continuous operator $\Pi_h : \Sigma^r \to \Sigma_h$ with the following properties. Given a discrete $z_h \in U_h$, we write $\phi = \phi(z_h)$ for the first component of the solution $(\phi, w) \in \Sigma^r \times U^r$ of $T(\phi, w) = (0, z_h)$. Assume that there are functions $\rho_\Sigma = \rho_\Sigma(h)$ and $\rho_X = \rho_X(h)$ with $\rho_\Sigma, \rho_X \to 0$ as $h \to 0$ and a Banach space $X \subset \Sigma_h' \cap (\Sigma^r)'$ such that

$$\text{(i)} \qquad b_h(\Pi_h\phi, v_h) = b(\phi, v_h) \qquad \text{for all } z_h \in U_h, v_h \in U_h,$$

$$\text{(ii)} \qquad \|\phi - \Pi_h\phi\|_H \lesssim \rho_\Sigma \|\phi\|_{\Sigma^r} \qquad \text{for all } z_h \in U_h,$$

$$\text{(iii)} \qquad |\langle g, \phi - \Pi_h\phi\rangle| \lesssim \rho_X \|g\|_X \|\phi\|_{\Sigma^r} \qquad \text{for all } z_h \in U_h, g \in X.$$

Assumption 6 (nonconformity control)**.** Given $\phi \in \Sigma^r$ and $w \in U^r$ satisfying $a(\phi, \tau) + b(\tau, w) = 0$ for all $\tau \in \Sigma$, we define the nonconforming consistency residual by

$$R_h(\tau_h; \phi, w) := a(\tau_h, \phi) + b_h(\tau_h, w) \quad \text{for all } \tau_h \in \Sigma_h$$

and assume that there exists a function $\rho_{\mathrm{nc}} = \rho_{\mathrm{nc}}(h)$ with $\rho_{\mathrm{nc}} \to 0$ as $h \to 0$ such that

$$|R_h(\tau_h; \phi, w)| \lesssim \rho_{\mathrm{nc}} \|\tau_h\|_{\Sigma,h} \left(\|\phi\|_{\Sigma^r} + \|w\|_{U^r}\right) \quad \text{for all } \tau_h \in \Sigma_h$$

and all such pairs (ϕ, w).

2.4. **Error estimate.** As our main theorem we give an abstract error estimate based on the continuous and discrete mixed formulations.

Theorem 1 (error estimate)**.** *Let Assumptions 1–6 be satisfied. Given $f \in U$ and $g \in X$, let $u \in U$ be the transposition solution defined in* (2.4) *and $(\sigma_h, u_h) \in \Sigma_h \times U_h$ be the solution of* (2.5)*. Then the following error estimate holds*

$$\|u - u_h\|_U \lesssim \inf_{v_h \in U_h} \|u - v_h\|_U + \rho_X \|g\|_X + \left(\rho_\Sigma + \rho_U + \rho_{\mathrm{nc}}\right) \left(\|f\|_U + \|g\|_{\Sigma_h'}\right).$$

We remark that the discrete norm of g itself on the right-hand side of the error estimate may diverge as $h \to 0$.

Proof of Theorem 1. The U-orthogonal projection $Q_h : U \to U_h$ satisfies

$$\|u - Q_h u\|_U = \inf_{v_h \in U_h} \|u - v_h\|_U \quad \text{and} \quad \|u - u_h\|_U^2 = \|u - Q_h u\|_U^2 + \|Q_h u - u_h\|_U^2.$$

Thus, setting $e_h = Q_h u - u_h \in U_h$, it remains to estimate $\|e_h\|_U$. Let $(\phi, w) \in \Sigma^r \times U^r$ be the regular adjoint pair determined by $T(\phi, w) = -(0, e_h)$. By (2.3) in Assumption 2, the solution satisfies

$$\|\phi\|_{\Sigma^r} + \|w\|_{U^r} \lesssim \|e_h\|_U. \tag{2.7}$$

Taking the second equation in $T(\phi, w) = -(0, e_h)$ with e_h gives

$$\|e_h\|_U^2 = (e_h, Q_h u - u_h)_U = (e_h, u - u_h)_U = (u, e_h)_U - b(\phi, u_h). \tag{2.8}$$

The definition of the transposition solution in (2.4) yields

$$(u, e_h)_U = \langle g, \phi\rangle - (f, w)_U.$$

The Fortin relation in Assumption 5(i) and the first discrete equation in (2.5) imply

$$b(\phi, u_h) = b_h(\Pi_h\phi, u_h) = \langle g, \Pi_h\phi\rangle - a(\sigma_h, \Pi_h\phi).$$

Substituting these two equalities into (2.8) gives

$$\|e_h\|_U^2 = \langle g, \phi - \Pi_h\phi\rangle + a(\sigma_h, \Pi_h\phi) - (f, w)_U. \tag{2.9}$$

Testing the second discrete equation in (2.5) with $P_h w$ gives

$$b_h(\sigma_h, P_h w) + (f, P_h w)_U = 0.$$

Adding this and $0 = R_h(\sigma_h;\phi,w) - a(\sigma_h,\phi) - b_h(\sigma_h,w)$ to (2.9) yields

$$\begin{aligned}\|e_h\|_U^2 = \langle g, \phi - \Pi_h\phi\rangle + a(\sigma_h, \Pi_h\phi - \phi) - (f, w - P_h w)_U \\ + R_h(\sigma_h;\phi,w) - b_h(\sigma_h, w - P_h w).\end{aligned}$$

Using the extension of $a(\cdot,\cdot)$ to $H \times H$ and $\Sigma, \Sigma_h \hookrightarrow H$, we infer from Assumption 5(ii)

$$|a(\sigma_h, \Pi_h\phi - \phi)| \lesssim \|\sigma_h\|_H \|\Pi_h\phi - \phi\|_H \lesssim \rho_\Sigma \|\sigma_h\|_H \|\phi\|_{\Sigma^r} \lesssim \rho_\Sigma \|\sigma_h\|_{\Sigma,h} \|\phi\|_{\Sigma^r}.$$

Continuity of b_h and Assumption 4 show

$$|b_h(\sigma_h, w - P_h w)| \le \|\sigma_h\|_{\Sigma,h} \|w - P_h w\|_U \lesssim \rho_U \|\sigma_h\|_{\Sigma,h} \|w\|_{U^r}.$$

This, combined with Assumptions 2 and 4–6, gives

$$\|e_h\|_U^2 \lesssim \Big(\rho_U \|f\|_U + \rho_X \|g\|_X + (\rho_\Sigma + \rho_U + \rho_{\mathrm{nc}}) \|\sigma_h\|_{\Sigma,h}\Big) \|e_h\|_U.$$

The discrete stability (2.6) completes the proof. □

Remark 2.4. Suppose that Assumption 5 holds more generally for all $z_h \in U$ rather than merely on the discrete subspace U_h. Then, setting $e_h := u - u_h$ and choosing (ϕ, w) as the regular adjoint pair satisfying $T(\phi, w) = -(0, e_h)$ directly in the proof of Theorem 1, the remaining arguments yield the following more explicit error estimate

$$\|u - u_h\|_U \lesssim \rho_X \|g\|_X + \big(\rho_\Sigma + \rho_U + \rho_{\mathrm{nc}}\big) \Big(\|f\|_U + \|g\|_{\Sigma_h'}\Big).$$

Except for the application to the Maxwell system, this stronger assumption is satisfied for all model problems considered in this paper. If, in addition, P_h satisfies $b_h(\tau_h, w - P_h w) = 0$, then we obtain the following error estimate

$$\|u - u_h\|_U \lesssim \rho_U \|f\|_U + \rho_X \|g\|_X + \big(\rho_\Sigma + \rho_{\mathrm{nc}}\big) \Big(\|f\|_U + \|g\|_{\Sigma_h'}\Big).$$

Remark 2.5 (conforming reduction). Suppose $\Sigma_h \subset \Sigma$ and b_h, b coincide. Then the discrete method (2.5) is conforming and the error estimate in Theorem 1 holds with $\rho_{\mathrm{nc}} = 0$.

Remark 2.6. For the transposition solution u, a stress-like variable can be recovered $\sigma^{\mathrm{tr}} \in (\Sigma^r)'$ through

$$\langle \sigma^{\mathrm{tr}}, \tau\rangle := \langle g, \tau\rangle - b(\tau, u) \qquad \text{for all } \tau \in \Sigma^r.$$

If we additionally assume that the three properties in Assumption 5 hold for arbitrary $\tau \in \Sigma^r$, then this definition, the first discrete equation in (2.5), and the Fortin relation in Assumption 5(i) give the identity

$$\langle \sigma^{\mathrm{tr}}, \tau\rangle - a(\sigma_h, \tau) = \langle g, \tau - \Pi_h\tau\rangle + a(\sigma_h, \Pi_h\tau - \tau) - b(\tau, u - u_h).$$

We then see

$$|\langle \sigma^{\mathrm{tr}}, \tau\rangle - a(\sigma_h, \tau)| \lesssim \Big(\|u - u_h\|_U + \rho_X \|g\|_X + \rho_\Sigma \|\sigma_h\|_{\Sigma,h}\Big) \|\tau\|_{\Sigma^r}.$$

Taking the supremum over nonzero $\tau \in \Sigma^r$ and substituting the error estimate of $\|u - u_h\|_U$ in Theorem 1 proves that the error that

$$\|\sigma^{\mathrm{tr}} - a(\sigma_h, \cdot)\|_{(\Sigma^r)'}$$

is controlled by the upper bound from Theorem 1.

Remark 2.7. The discretization parameter h can be localized in concrete applications where it often describes a local mesh size of a finite element triangulation. In that case, h is an L^∞ function that gives rise to well-defined weighted norms $\|\rho_U \cdot\|_U$ and so forth. For the sake of a simple presentation, we refrain from this formalization step and leave the usually simple generalization to particular FEM applications.

3. Application to the Laplacian

Let $\Omega \subseteq \mathbb{R}^m$ with $m = \{2, 3\}$ be an open, bounded, connected, polytopal Lipschitz domain. The Poisson equation with Dirichlet boundary condition reads

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{3.1}$$

We illustrate a discretization with low-order Raviart–Thomas FEM, but the error estimate applies to many other mixed schemes as well.

3.1. **Notation.** We briefly fix some standard notation that will apply throughout the remaining parts of this article. Given a finite-dimensional real vector space Y, let $H^\ell(G;Y)$ denote the standard Hilbertian Sobolev space of real order $\ell \geq 0$ on the open domain G, equipped with the norm $\|\cdot\|_{H^\ell(G)}$. In particular, $L^2(G;Y)$ denotes the space of square-integrable functions associated with the inner product $(\cdot,\cdot)_{L^2(G)}$. The subspace $H^\ell_0(G;Y)$ denotes the closure of $C^\infty_0(G;Y)$ in $H^\ell(G;Y)$, and $H^{-\ell}(G;Y)$ denotes the dual space of $H^\ell_0(G;Y)$. If $Y = \mathbb{R}$, we simply write $H^\ell(G)$, $L^2(G)$, and so forth. The space of real $m \times m$ matrices is denoted by $\mathbb{M}$, symmetric matrices are denoted by $\mathbb{S} := \{M \in \mathbb{M} : M = M^\top\}$, and the $m \times m$ unit matrix is denoted by I. For $0 < s < 1$ and an m-dimensional Lipschitz domain G, define $H^s(\partial G;Y) := \{g \in L^2(\partial G;Y) : \|g\|_{H^s(\partial G)} < \infty\}$, where

$$|g|^2_{H^s(\partial G)} := \int_{\partial G}\int_{\partial G} \frac{|g(x)-g(y)|^2_Y}{|x-y|^{m-1+2s}}\, ds_x\, ds_y,\ \|g\|^2_{H^s(\partial G)} := \|g\|^2_{L^2(\partial G)} + |g|^2_{H^s(\partial G)}.$$

It is assumed that the discrete spaces are defined over simplicial triangulation $\mathcal{T}_h$ of $\bar\Omega$ from a shape-regular family, that they are quasi-uniform, and generic constants may depend on their shape-regularity. We note that the results of this paper remain true on more general meshes that are only locally quasi-uniform because the critical estimates can be localized. However, for the sake of a simple exposition, we impose the quasi-uniformity condition here and mention that the proofs will generalize with minor modifications. For any element $K \in \mathcal{T}_h$ and degree $\ell \geq 0$, $P_\ell(K;Y)$ denotes the space of Y-valued polynomials of degree at most ℓ, with $P_\ell(\mathcal{T}_h;Y)$ denoting the associated piecewise polynomial space with respect to $\mathcal{T}_h$. Again, Y is omitted in the notation when $Y = \mathbb{R}$. The mesh size is denoted by $h := \max_{K\in\mathcal{T}_h} h_K$, where $h_K := \operatorname{diam}(K)$. Let $\mathcal{F}_h$, $\mathcal{E}_h$, and $\mathcal{V}_h$ denote the sets of hyperfaces, edges, and vertices in $\mathcal{T}_h$, respectively. Superscripts i and ∂ distinguish interior and boundary entities (e.g., $\mathcal{V}_h = \mathcal{V}_h^{\mathrm{i}} \cup \mathcal{V}_h^{\partial}$). In addition, the local sets of faces, edges, and vertices of an element $K \in \mathcal{T}_h$ are denoted by $\mathcal{F}(K)$, $\mathcal{E}(K)$, and $\mathcal{V}(K)$, respectively. The L^2-projection onto $P_\ell(\mathcal{T}_h)$ is denoted by Q^ℓ_h. Similarly, Q^ℓ_F and Q^ℓ_e represent the L^2-projections onto $P_\ell(F)$ and $P_\ell(e)$ for a face F and an edge e, respectively, and Q^ℓ_∂ denotes the piecewise boundary L^2-projection.

3.2. **Mixed method.** The mixed formulation of (3.1) is based on spaces

$$\Sigma = H(\operatorname{div},\Omega;\mathbb{R}^m) := \{\tau \in L^2(\Omega;\mathbb{R}^m) : \operatorname{div}\tau \in L^2(\Omega)\} \text{ and } U = L^2(\Omega),$$

with norms $\|\cdot\|_\Sigma = \|\cdot\|_{L^2(\Omega)} + \|\operatorname{div}\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_U = \|\cdot\|_{L^2(\Omega)}$. The bilinear forms are

$$a(\sigma,\tau) = (\sigma,\tau)_{L^2(\Omega)}, \quad b(\tau,v) = (\operatorname{div}\tau, v)_{L^2(\Omega)} \text{ for all } \sigma,\tau \in \Sigma, v \in U.$$

With $H = L^2(\Omega;\mathbb{R}^m)$ normed by $\|\cdot\|_H = \|\cdot\|_{L^2(\Omega)}$, $a(\cdot,\cdot)$ extends to a bounded, symmetric, positive semi-definite form on $H \times H$. Assumption 1 follows from classical theory [11]. Let $s \in (1/2, 1]$ be an admissible elliptic shift exponent for the homogeneous Poisson Dirichlet problem, i.e., a number such that $f \in U$ in (3.1) implies $\|u\|_{H^{1+s}(\Omega)} \lesssim \|f\|_U$. In two and three space dimensions, the existence of s is proven in [29, 20]. Let $U^r = H^{1+s}(\Omega) \cap H^1_0(\Omega)$ with the norm $\|\cdot\|_{U^r} = \|\cdot\|_{H^{1+s}(\Omega)}$ and

$$\Sigma^r = \Sigma \cap H^s(\Omega;\mathbb{R}^m) \text{ with the norm } \|\cdot\|_{\Sigma^r} = \|\cdot\|_{H^s(\Omega)} + \|\operatorname{div}\cdot\|_{L^2(\Omega)}.$$

With this choice of spaces, given any $z \in L^2(\Omega)$, the solution to $T(\phi, w) = (0, z)$ is unique and satisfies (2.3), see, e.g., [29, 20] or the summary result in [25, Lemma 3.4]. This regularity result proves Assumption 2.

Given $K \in \mathcal{T}_h$, we define the usual Raviart–Thomas space as $RT_0(K) := \{\tau : K \to \mathbb{R}^m : \tau(x) = \alpha + \beta x \text{ for } \alpha \in \mathbb{R}^m, \beta \in \mathbb{R}\}$ and the stable conforming lowest-order Raviart–Thomas pair

$$\Sigma_h = \{\tau \in \Sigma : \tau|_K \in RT_0(K) \text{ for all } K \in \mathcal{T}_h\} \quad \text{and} \quad U_h = P_0(\mathcal{T}_h).$$

Assumption 3 holds for this discrete pair [11]. Let $X = H^t(\partial\Omega)$ with $0 \le t < 1/2$ and $g \in X$. In the discrete formulation, we have $\langle g, \tau_h\rangle = (g, \tau_h \cdot n)_{L^2(\partial\Omega)}$ for all $\tau_h \in \Sigma_h$, where n denotes the outer unit normal vector of the boundary $\partial\Omega$. For $t = 0$, the Cauchy-Schwarz inequality and the inverse trace inequality give

$$|\langle g, \tau_h\rangle| \le \|g\|_{L^2(\partial\Omega)}\|\tau_h \cdot n\|_{L^2(\partial\Omega)} \lesssim h^{-1/2}\|g\|_{L^2(\partial\Omega)}\|\tau_h\|_{L^2(\Omega)}.$$

For $0 < t < 1/2$, there exists a lifting $G \in H^{1/2+t}(\Omega)$ such that $G|_{\partial\Omega} = g$ and $\|G\|_{H^{1/2+t}(\Omega)} \lesssim \|g\|_{H^t(\partial\Omega)}$. A generalized integration by parts gives

$$\langle g, \tau_h\rangle = (\operatorname{div}\tau_h, G)_{L^2(\Omega)} + \langle \nabla G, \tau_h\rangle_\Omega,$$

where the last term is the duality product between $H^{t-1/2}(\Omega)$ and $H^{1/2-t}(\Omega) = H^{1/2-t}_0(\Omega)$. The Cauchy-Schwarz inequality then implies

$$|(\operatorname{div}\tau_h, G)_{L^2(\Omega)}| \le \|\operatorname{div}\tau_h\|_{L^2(\Omega)}\|G\|_{L^2(\Omega)} \lesssim \|\tau_h\|_\Sigma \|g\|_{H^t(\partial\Omega)}.$$

The inverse estimate in fractional Sobolev spaces shows

$$\langle \tau_h, \nabla G\rangle_\Omega \le \|\nabla G\|_{H^{t-1/2}(\Omega)}\|\tau_h\|_{H^{1/2-t}(\Omega)} \lesssim h^{t-1/2}\|G\|_{H^{t+1/2}(\Omega)}\|\tau_h\|_{L^2(\Omega)}.$$

Thus, $\langle g, \tau_h\rangle_{\partial\Omega} \lesssim (1 + h^{t-1/2})\|g\|_{H^t(\partial\Omega)}\|\tau_h\|_\Sigma$. This, combined with the case $t = 0$, shows the a priori stability bound

$$\|\sigma_h\|_{\Sigma,h} + \|u_h\|_U \lesssim \|f\|_{L^2(\Omega)} + h^{t-1/2}\|g\|_X \text{ for all } 0 \le t < 1/2. \tag{3.2}$$

Let $P_h : U \to U_h$ be the L^2 projection onto piecewise constants. The Poincaré inequality implies

$$\|w - P_h w\|_{L^2(\Omega)} \lesssim h\|w\|_{U^r} \quad \text{for all } w \in U^r,$$

which verifies Assumption 4 with $\rho_U = h$. Let $\Pi_h : \Sigma^r \to \Sigma_h$ be the commuting Raviart–Thomas interpolation [11]. Since $\operatorname{div}\Pi_h\phi = P_h \operatorname{div}\phi$ for all $\phi \in \Sigma^r$, Assumption 5(i) holds. Furthermore,

$$\|\phi - \Pi_h\phi\|_{L^2(\Omega)} \lesssim h^s\|\phi\|_{\Sigma^r} \quad \text{for all } \phi \in \Sigma^r.$$

This gives Assumption 5(ii) with $\rho_\Sigma = h^s$. Since the Raviart–Thomas projection Π_h is defined such that $(\Pi_h\phi)\cdot n = Q^0_\partial(\phi \cdot n)$ holds for all $\phi \in \Sigma^r$ on $\partial\Omega$, we infer

$$\begin{aligned}|\langle g, (\phi - \Pi_h\phi)\cdot n\rangle| &\le \|g - Q^0_\partial g\|_{L^2(\partial\Omega)}\|(1 - Q^0_\partial)(\phi\cdot n)\|_{L^2(\partial\Omega)}\\ &\lesssim h^{s+t-1/2}\|g\|_{H^t(\partial\Omega)}\|\phi\|_{H^s(\Omega)}.\end{aligned}$$

Thus $\rho_X = h^{s+t-1/2}$. Since $\Sigma_h \subset \Sigma$, Remark 2.5 applies and $\rho_{\mathrm{nc}} = 0$. We collect these findings as the following result, implied by Theorem 1.



(A) Unit square. (B) L-shaped domain. (C) Unit cube. (D) L-shaped prism.

FIGURE 1. Initial meshes in two and three dimensions.

Corollary 3.1. Let $\Omega \subset \mathbb{R}^m$ be a bounded, open Lipschitz polytope and assume there exists some $1/2 < s \leq 1$ such that the homogeneous Dirichlet problem for the Laplacian satisfies elliptic $H^{1+s}(\Omega)$ regularity. Let $f \in L^2(\Omega)$ and $g \in H^t(\partial\Omega)$ with $0 \leq t < 1/2$. Then, the transposition solution u of the Laplacian $-\Delta u = f$ in Ω with boundary data g on $\partial\Omega$ and the lowest-order Raviart-Thomas solution (σ_h, u_h) to (2.5) satisfy the error bound

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^s \|f\|_{L^2(\Omega)} + h^{s+t-1/2} \|g\|_{H^t(\partial\Omega)}.$$

Remark 3.2. The L^2 error estimate of $u - u_h$ in Corollary 3.1 coincides with the result provided in [25, Theorem 2.2] and [25, Corollary 5.1]. The proof shown here can dispense with a regularization procedure.

Example 3.3. We introduce some notation that will be used in all numerical illustrations throughout this paper. We consider the planar domains Ω_S and Ω_L and their three-dimensional extrusions Ω_C and Ω_P, defined by

$$\begin{aligned} &\Omega_S = (0,1)^2, && \Omega_L = (-1/2,1/2)^2 \setminus \big([0,1/2] \times [-1/2,0]\big), \\ &\Omega_C = \Omega_S \times (0,1), && \Omega_P = \Omega_L \times (0,1). \end{aligned} \tag{3.3}$$

Thus, Ω_S is the unit square, Ω_L is an L-shaped domain, Ω_C is the unit cube, and Ω_P is an L-shaped prism. The distinguished corner is the origin in both planar domains; it is convex for Ω_S and reentrant for Ω_L. The corresponding edge in the three-dimensional domains is $\{(0,0,z) : 0 < z < 1\}$. The opening angles at these corners, or along these edges, are $\omega_S = \omega_C = \pi/2$ and $\omega_L = \omega_P = 3\pi/2$. We use polar coordinates (r,θ) in the (x,y)-plane, centered at the origin. The angular ranges are $0 < \theta < \omega_S$ on Ω_S and $0 < \theta < \omega_L$ on Ω_L. The same convention applies to the horizontal cross sections of Ω_C and Ω_P, respectively. The initial meshes are shown in Figure 1, and subsequent meshes are obtained by uniform refinement.

For the actual example in the context of the Poisson equation, we consider the three-dimensional domains Ω_C and Ω_P. We set $\alpha = -1/3$ and define the singular function ψ and the exact solution by

$$\psi(r,\theta) = r^\alpha \sin(\alpha\theta), \qquad u(x,y,z) = \psi(r,\theta) z(1-z). \tag{3.4}$$

The right-hand side is $f(x,y,z) = 2\psi(r,\theta)$. The induced boundary datum satisfies

$$g = u|_{\partial\Omega} \in H^{1/6-\nu}(\partial\Omega) \text{ for any } \nu > 0.$$

Given that the shift regularity indices for these domains are $s_C = 1$ and $s_P = 2/3$ [20, 29], the predicted L^2 convergence rates from Corollary 3.1 (with $t = 1/6 - \nu$) are (almost) $2/3$ and $1/3$, respectively. Table 1 displays the L^2 errors along with experimental convergence orders, which conform to these predictions. The evaluation of integrals involving singular functions is done via composite Gaussian quadrature with local subdivision near the singular edge. □

3.3. **The Crouzeix–Raviart method.** The nonconforming P_1 method due to Crouzeix–Raviart offers an alternative scheme that allows for a local L^2 approximation of irregular boundary data. It is well known [5, 33] that this discretization is in one-to-one correspondence with the mixed Raviart–Thomas method, and we

	Unit cube		L-shaped prism	
$h_\ell/\sqrt{3}$	$\|u-u_h\|_{L^2(\Omega)}$	order	$\|u-u_h\|_{L^2(\Omega)}$	order
2^{-1}	2.2668e−2	–	6.0814e−2	–
2^{-2}	1.3396e−2	0.7589	3.7741e−2	0.6883
2^{-3}	7.8087e−3	0.7786	2.4190e−2	0.6417
2^{-4}	4.6202e−3	0.7571	1.6439e−2	0.5573
2^{-5}	2.7804e−3	0.7327	1.1767e−2	0.4824
2^{-6}	1.6969e−3	0.7124	8.7462e−3	0.4280
2^{-7}	1.0466e−3	0.6973	6.6644e−3	0.3922
2^{-8}	6.5020e−4	0.6867	5.1577e−3	0.3697

TABLE 1. L^2 errors and observed convergence orders for the RT_0-P_0 mixed approximation of the three-dimensional Poisson problem with boundary data $g \in H^{1/6-\nu}(\partial\Omega)$.

can thereby derive an error bound for that scheme as a corollary of the previous result. We mention that, alternatively, a direct proof without the detour over the mixed method can be worked out as well.

For $g \in X$, let $V_{h,g}^{\mathrm{nc}}$ denote the Crouzeix–Raviart finite element space with boundary values prescribed by g. This affine space is spanned by the piecewise affine functions that satisfy $\int_F [v_h] ds = 0$ for each interior face $F \in \mathcal{F}_h^i$ (here brackets denote the interface jump) and $\int_F (g - v_h) ds = 0$ for each boundary face $F \in \mathcal{F}_h^\partial$. Then, $V_{h,0}^{\mathrm{nc}}$ denotes the corresponding homogeneous space. The nonconforming method seeks $u_h^{\mathrm{nc}} \in V_{h,g}^{\mathrm{nc}}$ such that

$$(\nabla_h u_h^{\mathrm{nc}}, \nabla_h v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} \qquad \text{for all } v_h \in V_{h,0}^{\mathrm{nc}}, \tag{3.5}$$

where ∇_h denotes the piecewise gradient with respect to $\mathcal{T}_h$. We introduce the auxiliary discrete solution $\tilde{u}_h^{\mathrm{nc}} \in V_{h,g}^{\mathrm{nc}}$ satisfying (3.5) with f replaced by $f_h = Q_h^0 f$.

Let $(\sigma_h, u_h) \in \Sigma_h \times U_h$ denote the RT_0–P_0 solution introduced in the previous subsection. The classical equivalence due to [33] states

$$\sigma_h = \nabla_h \tilde{u}_h^{\mathrm{nc}} - m^{-1} f_h r_h, \qquad u_h = Q_h^0 \tilde{u}_h^{\mathrm{nc}} + m^{-2} f_h Q_h^0(|r_h|^2). \tag{3.6}$$

where, for every $K \in \mathcal{T}_h$, $r_h|_K(x) := x - x_K$ with the barycenter x_K of K.

Corollary 3.4. Let the conditions of Corollary 3.1 hold. Then the transposition solution u of the Laplacian $-\Delta u = f$ in Ω with boundary data g on $\partial\Omega$ and the Crouzeix–Raviart solution to (3.5) satisfy the error bound

$$\|u - u_h^{\mathrm{nc}}\|_{L^2(\Omega)} \lesssim h^s \|f\|_{L^2(\Omega)} + h^{s+t-1/2} \|g\|_{H^t(\partial\Omega)}.$$

Proof. By the triangle inequality,

$$\|u - u_h^{\mathrm{nc}}\|_{L^2(\Omega)} \le \|u - u_h\|_{L^2(\Omega)} + \|u_h - \tilde{u}_h^{\mathrm{nc}}\|_{L^2(\Omega)} + \|\tilde{u}_h^{\mathrm{nc}} - u_h^{\mathrm{nc}}\|_{L^2(\Omega)}. \tag{3.7}$$

A triangle inequality, (3.6), and the element-wise Poincaré inequality give

$$\begin{aligned} \|u_h - \tilde{u}_h^{\mathrm{nc}}\|_{L^2(\Omega)} &\le \|u_h - Q_h^0 \tilde{u}_h^{\mathrm{nc}}\|_{L^2(\Omega)} + \|Q_h^0 \tilde{u}_h^{\mathrm{nc}} - \tilde{u}_h^{\mathrm{nc}}\|_{L^2(\Omega)} \\ &\lesssim h^2 \|f_h\|_{L^2(\Omega)} + h \|\nabla_h \tilde{u}_h^{\mathrm{nc}}\|_{L^2(\Omega)}. \end{aligned}$$

The combination of (3.6) and the stability result (3.2) shows

$$\|\nabla_h \tilde{u}_h^{\mathrm{nc}}\|_{L^2(\Omega)} \lesssim \|f_h\|_{L^2(\Omega)} + h^{t-1/2} \|g\|_{H^t(\partial\Omega)}.$$

This implies

$$\|u_h - \tilde{u}_h^{\mathrm{nc}}\|_{L^2(\Omega)} \lesssim h \|f\|_{L^2(\Omega)} + h^{t+1/2} \|g\|_{H^t(\partial\Omega)}.$$

Set $e_h := \tilde{u}_h^{\mathrm{nc}} - u_h^{\mathrm{nc}} \in V_{h,0}^{\mathrm{nc}}$. Subtracting the two versions of (3.5) with f and f_h on the right-hand side, respectively, and testing with e_h gives $\|\nabla_h e_h\|_{L^2(\Omega)}^2 =$

$(f_h - f, e_h)_{L^2(\Omega)} = (f_h - f, e_h - Q_h^0 e_h)_{L^2(\Omega)}$. Hence, the approximation property of Q_h^0 implies $\|\nabla_h e_h\|_{L^2(\Omega)} \lesssim h\|f - f_h\|_{L^2(\Omega)}$. The discrete Poincaré inequality [13, (10.6.14)] therefore yields

$$\|\tilde{u}_h^{\mathrm{nc}} - u_h^{\mathrm{nc}}\|_{L^2(\Omega)} \lesssim \|\nabla_h e_h\|_{L^2(\Omega)} \lesssim h\|f - f_h\|_{L^2(\Omega)} \lesssim h\|f\|_{L^2(\Omega)}.$$

Substituting the preceding estimates and the estimate in Corollary 3.1 into (3.7) and using $s \leq 1$ completes the proof. □

4. Application to the Stokes equations

Let Ω be a domain as in Section 3. We consider the stationary Stokes problem

$$-\Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \text{ in } \Omega, \quad \text{and} \quad u = g \text{ on } \partial\Omega, \tag{4.1}$$

assuming that g satisfies the compatibility condition $(g \cdot n, 1)_{L^2(\partial\Omega)} = 0$. Following [15], we introduce the pseudo-stress $\sigma = -pI + \nabla u$ as an auxiliary variable, which is not necessarily symmetric. The Stokes equation (4.1) can be rewritten as

$$\operatorname{Div} \sigma = -f, \quad \mathcal{A}\sigma - \nabla u = 0 \text{ in } \Omega, \quad \text{and} \quad u = g \text{ on } \partial\Omega,$$

where Div acts row-wise and $\mathcal{A}$ is the deviatoric tensor defined by $\mathcal{A}\tau = \tau - m^{-1} \operatorname{tr}\tau I$ for any matrix τ. The incompressibility constraint $\operatorname{div} u = \operatorname{tr}(\nabla u) = 0$ is enforced, and the pressure trace $\operatorname{tr}\sigma = -mp$ is uniquely determined by imposing $(\operatorname{tr}\sigma, 1)_{L^2(\Omega)} = 0$. The mixed formulation is based on the spaces

$$\Sigma = \{\tau \in H(\operatorname{Div}, \Omega; \mathbb{M}) : (\operatorname{tr}\tau, 1)_{L^2(\Omega)} = 0\}, \quad U = L^2(\Omega; \mathbb{R}^m)$$

with norms $\|\cdot\|_\Sigma = \|\cdot\|_{L^2(\Omega)} + \|\operatorname{Div}\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_U = \|\cdot\|_{L^2(\Omega)}$. The bilinear forms are

$$a(\sigma, \tau) = (\mathcal{A}\sigma, \tau)_{L^2(\Omega)}, \quad b(\tau, v) = (\operatorname{Div}\tau, v)_{L^2(\Omega)} \text{ for all } \sigma, \tau \in \Sigma, v \in U.$$

We take $H = L^2(\Omega; \mathbb{M})$. Assumption 1 follows from the pseudostress formulation of the Stokes problem [15]. Let $s \in (1/2, 1]$ be an admissible elliptic shift exponent for the homogeneous Dirichlet problem of the Stokes equation. Its existence is shown in [29, 21], see also [26] and [1, Section 2.2]. Let $U^r = H^{1+s}(\Omega; \mathbb{R}^m) \cap H_0^1(\Omega; \mathbb{R}^m)$ with the norm $\|\cdot\|_{U^r} = \|\cdot\|_{H^{1+s}(\Omega)}$ and

$$\Sigma^r = \Sigma \cap H^s(\Omega; \mathbb{M}) \text{ with the norm } \|\cdot\|_{\Sigma^r} = \|\cdot\|_{H^s(\Omega)} + \|\operatorname{Div}\cdot\|_{L^2(\Omega)}.$$

Under these assumptions, given $z \in L^2(\Omega; \mathbb{R}^m)$, the solution to $T(\phi, w) = (0, z)$ is unique and satisfies (2.3).

Let $\mathcal{T}_h$ be a simplicial mesh as in the last section. Let $\widehat{\Sigma}_h$ denote the space of matrix-valued functions whose rows belong to the divergence-conforming Raviart–Thomas space described in Section 3. We consider the conforming discretizations as

$$\Sigma_h = \{\tau_h \in \widehat{\Sigma}_h : (\operatorname{tr}\tau_h, 1)_{L^2(\Omega)} = 0\} \quad \text{and} \quad U_h = P_0(\mathcal{T}_h; \mathbb{R}^m).$$

The stability of this pairing is shown in [15]. Let $X = \{g \in H^t(\partial\Omega; \mathbb{R}^m) : (g \cdot n, 1)_{L^2(\partial\Omega)} = 0\}$ with $0 \leq t < 1/2$. For $g \in X$ we define $\langle g, \tau\rangle = (g, \tau n)_{L^2(\partial\Omega)}$. By the boundary-trace estimate established in the context of (3.2), applied row-wise,

$$\|\sigma_h\|_\Sigma + \|u_h\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} + h^{t-1/2}\|g\|_{H^t(\partial\Omega)}.$$

Let $P_h : U \to U_h$ denote the L^2-orthogonal projection, which verifies Assumption 4 with $\rho_U = h$. Let $\widehat{\Pi}_h$ be the row-wise commuting Raviart–Thomas projection. To preserve the normalization of Σ_h, define

$$\Pi_h\phi = \widehat{\Pi}_h\phi - (m \operatorname{meas}(\Omega))^{-1}(\operatorname{tr}\widehat{\Pi}_h\phi, 1)_{L^2(\Omega)} I \quad \text{for all } \phi \in \Sigma^r.$$

Since the correction is a constant multiple of the identity matrix, it has zero row-wise divergence. Therefore $\operatorname{Div}\Pi_h\phi = P_h \operatorname{Div}\phi$, and hence Assumption 5(i) holds.

Furthermore, as in Section 3, Assumption 5(ii) holds with $\rho_\Sigma = h^s$. The compatibility condition $(g\cdot n, 1)_{L^2(\partial\Omega)} = 0$ gives $\langle g, (\phi - \Pi_h\phi)n\rangle = \langle g, (\phi - \widehat{\Pi}_h\phi)n\rangle$. Then, following similar procedures for deriving Assumption 5(iii) in the application of the Laplace equation, one can prove the following estimate

$$|\langle g, (\phi - \Pi_h\phi)n\rangle| \lesssim h^{s+t-1/2}\|g\|_{H^t(\partial\Omega)}\|\phi\|_{\Sigma^r}.$$

Therefore $\rho_X = h^{s+t-1/2}$. Since $\Sigma_h \subset \Sigma$, Remark 2.5 applies and $\rho_{\mathrm{nc}} = 0$.

Corollary 4.1. Let $\Omega \subset \mathbb{R}^m$ be a bounded, open Lipschitz polytope. Assume that, for some $1/2 < s \le 1$, the homogeneous Dirichlet Stokes problem with L^2 right-hand sides satisfies $H^{1+s}(\Omega;\mathbb{R}^m)$ regularity for the velocity and $H^s(\Omega)$ regularity for the pressure normalized to have zero mean. Let $f \in L^2(\Omega;\mathbb{R}^m)$ and $g \in H^t(\partial\Omega;\mathbb{R}^m)$ with $0 \le t < 1/2$, and assume $\int_{\partial\Omega} g\cdot n\, ds = 0$. Let u be the transposition velocity solution of the Stokes equation (4.1) and let (σ_h, u_h) be the lowest-order Raviart–Thomas mixed approximation in the pseudostress–velocity formulation. Then

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^s\|f\|_{L^2(\Omega)} + h^{s+t-1/2}\|g\|_{H^t(\partial\Omega)}.$$

Remark 4.2. For convex polygonal domains in two dimensions we have $s = 1$. This Corollary gives $\|u - u_h\|_{L^2(\Omega)} \lesssim h\|f\|_{L^2(\Omega)} + h^{1/2}\|g\|_{L^2(\partial\Omega)}$ for $g \in L^2(\partial\Omega;\mathbb{R}^m)$, which agrees with the optimal very weak Stokes estimates in [1]. We note that our error bound can dispense with logarithmic factors, which in [22] are a consequence of regularizing the boundary data.

Remark 4.3. Let $V_{h,g}^{\mathrm{nc}}, V_{h,0}^{\mathrm{nc}}$ be the vector-valued Crouzeix–Raviart space defined as in Section 3.3. The discrete pressure space is $U_h^0 = P_0(\mathcal{T}_h) \cap L_0^2(\Omega)$. The nonconforming CR–P_0 mixed method for the Stokes equation reads: find $(u_h^{\mathrm{nc}}, p_h^{\mathrm{nc}}) \in V_{h,g}^{\mathrm{nc}} \times U_h^0$ such that

$$\begin{cases} (\nabla_h u_h^{\mathrm{nc}}, \nabla_h v_h)_{L^2(\Omega)} - (p_h^{\mathrm{nc}}, \mathrm{div}_h\, v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} & \text{for all } v_h \in V_{h,0}^{\mathrm{nc}}, \\ (\mathrm{div}_h\, u_h^{\mathrm{nc}}, q_h)_{L^2(\Omega)} = 0 & \text{for all } q_h \in U_h^0. \end{cases}$$

Following similar procedures as in the proof in Section 3.3, one can obtain

$$\|u - u_h^{\mathrm{nc}}\|_{L^2(\Omega)} \lesssim h^s\|f\|_{L^2(\Omega)} + h^{s+t-1/2}\|g\|_{H^t(\partial\Omega)}.$$

Example 4.4. For the two-dimensional Stokes problem, we use the domains Ω_S and Ω_L and the associated mesh families introduced in Example 3.3. Set $\alpha = -1/3$ and $\omega = \omega_S$ or ω_L. The exact velocity and pressure are defined by

$$u(r,\theta) = r^\alpha(\Phi_1(\theta), \Phi_2(\theta))^T, \qquad \tilde{p}(r,\theta) = r^{\alpha-1}\Phi_p(\theta),$$

where the pressure is normalized by $p = \tilde{p} - \int_\Omega \tilde{p}\, dx/\,\mathrm{meas}(\Omega)$ and

$$\begin{aligned} \Phi_1(\theta) = {} & -\sin(\alpha\theta)\cos\omega - \alpha\sin\theta\cos\big(\alpha(\omega-\theta)+\theta\big) \\ & + \alpha\sin(\omega-\theta)\cos(\alpha\theta-\theta) + \sin\big(\alpha(\omega-\theta)\big), \\ \Phi_2(\theta) = {} & -\sin(\alpha\theta)\sin\omega - \alpha\sin\theta\sin\big(\alpha(\omega-\theta)+\theta\big) - \alpha\sin(\omega-\theta)\sin(\alpha\theta-\theta), \\ \Phi_p(\theta) = {} & 2\alpha\left[\sin\big((\alpha-1)\theta+\omega\big) + \sin\big((\alpha-1)\theta-\alpha\omega\big)\right]. \end{aligned}$$

The functions defined above satisfy $-\Delta u + \nabla p = 0$ and $\mathrm{div}\, u = 0$ in Ω. Since u behaves like $r^{-1/3}$ near the origin, its trace g belongs to $H^{1/6-\nu}\big(\partial\Omega;\mathbb{R}^2\big)$ for all $\nu > 0$. Table 2 displays the velocity L^2-errors and convergence rates. On the unit square ($s_S = 1$), the convergence rates are close to the theoretical order 2/3, while on the L-shaped domain they are close to the predicted rate $s_L + t - 1/2 = 0.211\ldots$ (with $s_L = 0.544\ldots$ and $t = 1/6 - \nu$). □

	Unit square		L-shaped domain	
$h_\ell/\sqrt{2}$	$\|u-u_h\|_{L^2(\Omega)}$	order	$\|u-u_h\|_{L^2(\Omega)}$	order
2^{-1}	2.3733e−1	–	1.9129e−1	–
2^{-2}	1.6508e−1	0.5238	1.3947e−1	0.4558
2^{-3}	1.0903e−1	0.5984	1.0525e−1	0.4061
2^{-4}	7.0175e−2	0.6357	8.1780e−2	0.3640
2^{-5}	4.4760e−2	0.6488	6.5123e−2	0.3286
2^{-6}	2.8414e−2	0.6556	5.3082e−2	0.2949
2^{-7}	1.7985e−2	0.6598	4.4137e−2	0.2662
2^{-8}	1.1364e−2	0.6624	3.7242e−2	0.2451

TABLE 2. L^2 errors and observed convergence orders for the pseudo-stress RT_0-P_0 approximation of the two-dimensional Stokes problem with boundary data $g \in H^{1/6-\nu}(\partial\Omega)$.

5. Application to linear elasticity

Let Ω be a domain as in Section 3. We consider the linear elasticity problem

$$-\operatorname{Div} \mathcal{C}\varepsilon(u) = f \quad \text{in } \Omega \qquad \text{and} \qquad u = g \quad \text{on } \partial\Omega.$$

Here $\varepsilon(u) = 1/2(\nabla u + (\nabla u)^T)$ is the symmetric part of the derivative matrix, and the elasticity tensor $\mathcal{C}\tau = 2\mu\tau + \lambda \operatorname{tr}\tau I$ for any symmetric matrix τ with material parameters $\mu, \lambda > 0$. The Hellinger–Reissner formulation is based on the spaces

$$\Sigma = H(\operatorname{Div}, \Omega; \mathbb{S}) := \{\tau \in L^2(\Omega; \mathbb{S}) : \operatorname{Div}\tau \in L^2(\Omega; \mathbb{R}^m)\}, \quad U = L^2(\Omega; \mathbb{R}^m)$$

equipped with norms $\|\cdot\|_\Sigma$ and $\|\cdot\|_U$ as in Section 4. Define the bilinear forms

$$a(\sigma, \tau) = (\mathcal{C}^{-1}\sigma, \tau)_{L^2(\Omega)}, \quad b(\tau, v) = (\operatorname{Div}\tau, v)_{L^2(\Omega)} \text{ for all } \sigma, \tau \in \Sigma, v \in U.$$

We let $H = L^2(\Omega; \mathbb{S})$. Standard theory [7, 11] implies the well-posedness of the saddle-point formulation (2.1). Let $s \in (1/2, 1]$ be an admissible regularity-shift exponent for the homogeneous Dirichlet elasticity problem. In two space dimensions, the existence of such s is shown in [29, 28]. We define $U^r = H^{1+s}(\Omega; \mathbb{R}^m) \cap H_0^1(\Omega; \mathbb{R}^m)$ with $\|\cdot\|_{U^r} = \|\cdot\|_{H^{1+s}(\Omega)}$, and

$$\Sigma^r = \Sigma \cap H^s(\Omega; \mathbb{S}) \text{ with the norm } \|\cdot\|_{\Sigma^r} = \|\cdot\|_{H^s(\Omega)} + \|\operatorname{Div}\cdot\|_{L^2(\Omega)}.$$

In three dimensions, the analysis below applies under the assumption that a corresponding regularity result holds. Under this hypothesis on s, Assumption 2 is satisfied.

Symmetric conforming mixed finite elements for tensor-valued problems typically require high-order polynomial spaces, and the regularity required for their full approximation order is not satisfied for irregular boundary data. This motivates the use of low-order nonconforming mixed finite element methods in the present setting. As an example, we explicitly describe the two-dimensional reduced Arnold–Winther element from [8]. For $K \in \mathcal{T}_h$, let $RM(K)$ denote the space of rigid-body motions as functions over K, i.e., linear combinations of constant vector fields and $(-x_2, x_1)$. For every interior face F, fix a unit normal n_F, and denote by $[\tau_h n_F]$ the corresponding jump. Define the discrete spaces

$$\Sigma_h = \left\{ \tau_h \in \Sigma(\mathcal{T}_h) : \int_F [\tau_h n_F] \cdot q \, ds = 0 \text{ for all } q \in P_1(F; \mathbb{R}^2), F \in \mathcal{F}_h^i \right\}$$

with

$$\Sigma(K) := \left\{ \tau \in P_2(K; \mathbb{S}) : n^T \tau n|_F \in P_1(F) \text{ for every } F \in \mathcal{F}(K), \operatorname{Div}\tau \in RM(K) \right\}$$

and $U_h = RM(\mathcal{T}_h)$, the piecewise rigid-body motions. A three-dimensional analogue is given in [4]. The subsequent analysis is formulated uniformly for $m = 2, 3$ and relies only on the structural properties shared by these elements. Throughout this section, F denotes an $(m-1)$-dimensional mesh facet.

Since $\Sigma_h \not\subset H(\mathrm{Div}, \Omega; \mathbb{S})$, we use the piecewise action Div_h of the divergence and set

$$b_h(\tau, v) = (\mathrm{Div}_h \tau, v)_{L^2(\Omega)}, \quad \text{for all } \tau \in \Sigma + \Sigma_h, v \in U.$$

The discrete graph norm is $\|\tau_h\|_{\Sigma,h}^2 := \|\tau_h\|_{L^2(\Omega)}^2 + \|\mathrm{Div}_h \tau_h\|_{L^2(\Omega)}^2$ for any $\tau_h \in \Sigma_h$. The stability of this pair is established in [8, 4]. Let $X = H^t(\partial\Omega; \mathbb{R}^m)$ with $0 \le t < 1/2$. For $g \in X$, define the discrete boundary functional by $\langle g, \tau_h \rangle = (g, \tau_h n)_{L^2(\partial\Omega)}$ for $\tau_h \in \Sigma_h$. For $t = 0$, $|\langle g, \tau_h \rangle| \le \|g\|_{L^2(\partial\Omega)} \|\tau_h n\|_{L^2(\partial\Omega)} \lesssim h^{-1/2} \|g\|_{L^2(\partial\Omega)} \|\tau_h\|_{L^2(\Omega)}$. For $0 < t < 1/2$, there exists a boundary lifting $G \in H^{1/2+t}(\Omega; \mathbb{R}^m)$ such that $G|_{\partial\Omega} = g$ and $\|G\|_{H^{1/2+t}(\Omega)} \lesssim \|g\|_{H^t(\partial\Omega)}$. An integration by parts gives

$$\langle g, \tau_h \rangle = (\mathrm{Div}_h \tau_h, G)_{L^2(\Omega)} + \langle \tau_h, \varepsilon(G) \rangle_\Omega - \sum_{F \in \mathcal{F}_h^i} ([\tau_h n_F], G)_{L^2(F)}. \tag{5.1}$$

Here, $\langle \tau_h, \varepsilon(G) \rangle_\Omega$ is the duality product between $H^{1/2-t}(\Omega; \mathbb{S})$ and $H^{t-1/2}(\Omega; \mathbb{S})$. The first term can be estimated by the Cauchy-Schwarz inequality

$$(\mathrm{Div}_h \tau_h, G)_{L^2(\Omega)} \le \|\mathrm{Div}_h \tau_h\|_{L^2(\Omega)} \|G\|_{L^2(\Omega)} \le \|g\|_{H^t(\partial\Omega)} \|\tau_h\|_{\Sigma,h}.$$

For the second term, the inverse estimate shows

$$\langle \tau_h, \varepsilon(G) \rangle_\Omega \le \|\tau_h\|_{H^{1/2-t}(\Omega)} \|\varepsilon(G)\|_{H^{t-1/2}(\Omega)} \lesssim h^{t-1/2} \|g\|_{H^t(\partial\Omega)} \|\tau_h\|_{L^2(\Omega)}.$$

By the discrete continuity conditions on Σ_h, it holds that

$$([\tau_h n_F], G)_{L^2(F)} = ([\tau_h n_F], G - Q_F^1 G)_{L^2(F)} \le \|[\tau_h n_F]\|_{L^2(F)} \|G - Q_F^1 G\|_{L^2(F)}.$$

The trace, inverse, and interpolation error estimates prove $([\tau_h n_F], G)_{L^2(F)} \lesssim h^{t-1/2} \|g\|_{H^t(\partial\Omega)} \|\tau_h\|_{\Sigma,h}$. Substituting these estimates into (5.1) and combining with the case $t = 0$ shows

$$\|\sigma_h\|_{\Sigma,h} + \|u_h\|_U \lesssim \|f\|_{L^2(\Omega)} + h^{t-1/2} \|g\|_X.$$

Let $P_h : U \to U_h$ denote the L^2-orthogonal projection, which satisfies Assumption 4 with $\rho_U = h$. We introduce the canonical interpolation operator $\Pi_h : \Sigma^r \to \Sigma_h$ defined locally by the degrees of freedom

$$\int_F \big((\phi - \Pi_h \phi) n_F\big) \cdot q \, ds = 0 \qquad \text{for all } q \in P_1(F; \mathbb{R}^m), F \in \mathcal{F}_h. \tag{5.2}$$

Then, $\mathrm{Div}_h \Pi_h \phi = P_h \mathrm{Div}\, \phi$ for all $\phi \in \Sigma^r$, which implies Assumption 5(i). The approximation estimate for the reduced interpolation operator follows from the scaling argument in [8, Section 5] and interpolation

$$\|\phi - \Pi_h \phi\|_{L^2(\Omega)} \lesssim h^s \|\phi\|_{\Sigma^r},$$

and hence Assumption 5(ii) holds with $\rho_\Sigma = h^s$. By the definition of Π_h in (5.2), it follows

$$\langle g, \phi - \Pi_h \phi \rangle = \sum_{F \in \mathcal{F}_h^\partial} \int_F (g - Q_F^1 g) \cdot (\phi - \Pi_h \phi) n \, ds \lesssim h^{s+t-1/2} \|g\|_{H^t(\partial\Omega)} \|\phi\|_{\Sigma^r}.$$

Thus Assumption 5(iii) holds with $\rho_X = h^{s+t-1/2}$. It remains to verify the nonconforming consistency condition. Let $(\phi, w) \in \Sigma^r \times U^r$ solve $a(\phi, \tau) + b(\tau, w) = 0$

for all $\tau \in \Sigma$. Since $\mathcal{C}^{-1}\phi = \varepsilon(w)$, element-wise integration by parts, $w = 0$ on $\partial\Omega$, and the moment continuity of $\tau_h n_F$ give for R_h defined in Assumption 6 that

$$R_h(\tau_h; \phi, w) = \sum_{F \in \mathcal{F}_h^i} \int_F [\tau_h n_F] \cdot w \, ds = \sum_{F \in \mathcal{F}_h^i} \int_F [\tau_h n_F] \cdot (w - Q_F^1 w) \, ds.$$

The trace approximation estimate and the polynomial inverse trace inequality give

$$|R_h(\tau_h; \phi, w)| \lesssim h^s \|\tau_h\|_{\Sigma,h} \|w\|_{U^r}.$$

Thus Assumption 6 holds with $\rho_{\mathrm{nc}} = h^s$.

Corollary 5.1. Let $\Omega \subset \mathbb{R}^m$ be an open and bounded Lipschitz polytope and assume that, for some $1/2 < s \leq 1$, the homogeneous Dirichlet elasticity problem with L^2 right-hand sides satisfies an $H^{1+s}(\Omega; \mathbb{R}^m)$ regularity assumption. Let $f \in L^2(\Omega; \mathbb{R}^m)$ and $g \in H^t(\partial\Omega; \mathbb{R}^m)$ with $0 \leq t < 1/2$. Then the transposition solution u of the linear elasticity problem and the reduced Arnold–Winther solution (σ_h, u_h) satisfy

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^s \|f\|_{L^2(\Omega)} + h^{s+t-1/2} \|g\|_{H^t(\partial\Omega)}.$$

Example 5.2. We consider the two-dimensional linear elasticity problem on the domains Ω_S and Ω_L from Example 3.3. The Lamé parameters are fixed as $\lambda = 10^6$ and $\mu = 1$. The exact displacement field is constructed from a harmonic-gradient potential:

$$\phi(r, \theta) = r^{1+\alpha} \sin\big((1+\alpha)\theta + \pi/4\big), \qquad u = \nabla\phi, \qquad \alpha = -1/3.$$

The body force vanishes ($f = 0$), and the Dirichlet condition $g = u|_{\partial\Omega}$ is prescribed on the entire boundary. It satisfies $g \in H^{1/6-\nu}(\partial\Omega; \mathbb{R}^2)$ for any $\nu > 0$. On the unit square Ω_S, the homogeneous Dirichlet dual problem admits the regularity shift $s_S = 1$. It is known [35] that on the L-shaped domain Ω_L with its reentrant angle $\omega_L = 3\pi/2$, the leading corner exponent κ_1 is the smallest positive root of

$$(\lambda + \mu)^{-2}(\lambda + 3\mu)^2 \sin^2(\kappa\omega_L) - \kappa^2 \sin^2 \omega_L = 0.$$

For the chosen Lamé parameters, $\kappa_1 = 0.544\ldots$, and every s_L with $1/2 < s_L < \kappa_1$ is an admissible regularity-shift exponent. Consequently, Corollary 5.1 yields the convergence orders $h^{2/3-\nu}$ and $h^{\kappa_1 - 1/3 - \nu}$ for any $\nu > 0$ for Ω_S and Ω_L, respectively, where $\kappa_1 - 1/3 = 0.211\ldots$. Table 3 displays the displacement L^2-errors and observed convergence orders for the reduced nonconforming Arnold–Winther method. On Ω_S, the observed orders approach the reference value $2/3$. On Ω_L, the orders on the finer meshes decrease towards the reference value $0.211\ldots$. These results are consistent with the reduced convergence rate associated with the reentrant corner. □

6. Application to the biharmonic equation

Let $\Omega \subseteq \mathbb{R}^2$ be an open, bounded, simply-connected, Lipschitz polygon. In this section, we only consider the planar case. We consider the biharmonic equation with clamped boundary conditions

$$\Delta^2 u = f \quad \text{in } \Omega \quad u = g_1, \quad \partial_n u = g_2 \text{ on } \partial\Omega, \tag{6.1}$$

where ∂_n is the normal derivative along $\partial\Omega$ and Δ^2 is the biharmonic operator.

The mixed formulation is based on the moment variable $\sigma = -\nabla^2 u$ and the spaces

$$\Sigma = H(\mathrm{divDiv}, \Omega; \mathbb{S}) := \{\tau \in L^2(\Omega; \mathbb{S}) : \mathrm{divDiv}\, \tau \in L^2(\Omega)\} \text{ and } U = L^2(\Omega)$$

	Unit square		L-shaped domain	
$h_\ell/\sqrt{2}$	$\|u-u_h\|_{L^2(\Omega)}$	order	$\|u-u_h\|_{L^2(\Omega)}$	order
2^{-1}	1.2419e−1	–	2.0601e−1	–
2^{-2}	8.0908e−2	0.6182	1.6149e−1	0.3512
2^{-3}	5.1950e−2	0.6391	1.1902e−1	0.4403
2^{-4}	3.3111e−2	0.6498	8.8904e−2	0.4209
2^{-5}	2.1010e−2	0.6562	6.8541e−2	0.3753
2^{-6}	1.3295e−2	0.6602	5.4658e−2	0.3265
2^{-7}	8.3989e−3	0.6626	4.4836e−2	0.2858
2^{-8}	5.3003e−3	0.6641	3.7529e−2	0.2566
2^{-9}	3.3427e−3	0.6651	3.1826e−2	0.2378

TABLE 3. L^2 displacement errors and observed convergence orders for the reduced nonconforming Arnold–Winther approximation of the two-dimensional linear elasticity problem with $\lambda = 10^6$, $\mu = 1$ and boundary data $g \in H^{1/6-\nu}(\partial\Omega)$.

with norms $\|\cdot\|_\Sigma = \|\cdot\|_{L^2(\Omega)} + \|\operatorname{divDiv}\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_U = \|\cdot\|_{L^2(\Omega)}$, respectively. Define the bilinear forms

$$a(\sigma,\tau) = (\sigma,\tau)_{L^2(\Omega)}, \quad b(\tau,v) = (\operatorname{divDiv}\tau, v) \text{ for all } \sigma,\tau\in\Sigma, v\in U.$$

We let $H = L^2(\Omega;\mathbb{S})$ with the L^2 norm. The well-posedness of this mixed formulation follows from [16]. It is known [10, 29] that the clamped biharmonic problem over a polygon with homogeneous boundary data possesses $H^{2+s}(\Omega)$ regularity with some $1/2 < s \le 2$, if $f \in L^2(\Omega)$. We refer to s as an admissible regularity index for the homogeneous clamped biharmonic problem and define $U^r = H^{2+s}(\Omega)\cap H_0^2(\Omega)$ with $\|\cdot\|_{U^r} = \|\cdot\|_{H^{2+s}(\Omega)}$, and

$$\Sigma^r = \Sigma\cap H^s(\Omega;\mathbb{S}) \text{ with the norm } \|\cdot\|_{\Sigma^r} = \|\cdot\|_{H^s(\Omega)} + \|\operatorname{divDiv}\cdot\|_{L^2(\Omega)}.$$

Then, for every $z \in L^2(\Omega)$, the solution to $T(\phi,w) = (0,z)$ is unique and satisfies (2.3), which implies Assumption 2.

In this section, the regularity of traces will be described by the symbols t_1, t_2. Therefore, we can denote the unit tangent vector to $\partial\Omega$ by $t = (-n_2, n_1)^T$ without conflicting notation. In this section, we will use the space $H^1(\partial\Omega)$ with norm $\|v\|^2_{H^1(\partial\Omega)} = \|v\|^2_{L^2(\partial\Omega)} + \|\partial_t v\|^2_{L^2(\partial\Omega)}$ and $H^\nu(\partial\Omega)$ for $1 < \nu < 3/2$, with norm $\|v\|^2_{H^\nu(\partial\Omega)} = \|v\|^2_{H^1(\partial\Omega)} + |\partial_t v|^2_{H^{\nu-1}(\partial\Omega)}$ for the boundary of the planar Lipschitz polygon Ω. It is proven in [6, Theorem 6.1] that for these ν the space coincides with the trace space of $H^{1/2+\nu}(\Omega)$. For a scalar function a and vector function q, define the following rotated-gradient (vector curl) operations in the plane

$$\nabla^\perp a = (\partial_2 a, -\partial_1 a)^T, \quad \nabla^\perp q = \begin{pmatrix} \partial_2 q_1 & -\partial_1 q_1 \\ \partial_2 q_2 & -\partial_1 q_2 \end{pmatrix}, \quad \nabla^\perp_{\mathbb{S}} q = \operatorname{sym}(\nabla^\perp q).$$

In the following, we consider the conforming triangle element proposed by Führer and Heuer [24]. Three-dimensional conforming elements can be found in [17, 30], but are not discussed here. Let $\mathcal{T}_h$ be a triangulation of Ω as in prior sections. Following [24], let $U_h = P_1(\mathcal{T}_h)$ and

$$\Sigma_h = \{\tau_h \in H(\operatorname{divDiv},\Omega;\mathbb{S}) : \tau_h|_K \in \nabla^\perp_{\mathbb{S}}(RT_2(K)) \oplus xx^T P_1(K) \text{ for all } K\in\mathcal{T}_h\},$$

$RT_2(K) = P_2(K;\mathbb{R}^2) + xP_2(K)$ denotes the local Raviart–Thomas space of second order [11]. The dimension of the local shape function space for Σ_h is 15. The pair (Σ_h, U_h) is uniformly inf-sup stable [24] and, thus, Assumption 3 is satisfied. Let $X := H^{t_1}(\partial\Omega)\times H^{t_2}(\partial\Omega)$ with $1 \le t_1 < 3/2$ and $0 \le t_2 < 1/2$. Due to the low

regularity of $\phi \in \Sigma^r$, the definition of the trace pairing and the construction of the interpolation operator are not obvious. For this, we introduce the following lemma.

We note that the kernel of the symmetric curl equals the three-dimensional space RT in the plane, spanned by the constant vector fields and the identity mapping x. In what follows we denote the space of H^1 fields that are L^2-orthogonal to RT by $H^1(\Omega;\mathbb{R}^2)/RT$.

Lemma 6.1 (regular decomposition)**.** *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, simply-connected Lipschitz polygon. Every $\tau \in \Sigma$ admits a unique decomposition $\tau = pI + \nabla_{\mathbb{S}}^{\perp} q$, with $p \in H_0^1(\Omega)$ and $q \in H^1(\Omega;\mathbb{R}^2)/RT$. Moreover, $p \in H^{3/2+\varepsilon}(\Omega)$ for the $0 < \varepsilon \leq 1/2$ describing the elliptic regularity of the Laplacian on the polygon Ω. The functions p and q depend linearly on τ and satisfy*

$$\|p\|_{H^{3/2+\varepsilon}(\Omega)} \lesssim \|\operatorname{divDiv}\tau\|_{L^2(\Omega)}, \qquad \|q\|_{H^1(\Omega)} \lesssim \|\tau\|_{\Sigma}.$$

If additionally $\tau \in H^{\alpha}(\Omega;\mathbb{S})$ for some $0 \leq \alpha \leq 3/2+\varepsilon$, then $q \in H^{1+\alpha}(\Omega;\mathbb{R}^2)$ and it satisfies

$$\|q\|_{H^{1+\alpha}(\Omega)} \lesssim \|\tau\|_{H^{\alpha}(\Omega)} + \|\operatorname{divDiv}\tau\|_{L^2(\Omega)}. \tag{6.2}$$

If τ additionally belongs to $H^s(\Omega;\mathbb{S})$ with $s > 1/2$ or τ is piecewise polynomial, then $q|_{\partial\Omega} \in H^1(\partial\Omega;\mathbb{R}^2)$.

Proof. The proof follows ideas from [31, Theorem 3.1]. Let $p \in H_0^1(\Omega)$ denote the solution to the Poisson problem $\Delta p = \operatorname{divDiv}\tau$. Elliptic regularity on polygons [29] shows, for some $0 < \varepsilon \leq 1/2$, that $\|p\|_{H^{3/2+\varepsilon}(\Omega)} \lesssim \|\operatorname{divDiv}\tau\|_{L^2(\Omega)}$. Furthermore, $\eta := \tau - pI$ satisfies $\operatorname{divDiv}\eta = 0$. If $\eta \in H^{\alpha}(\Omega;\mathbb{S})$, [19, Theorem 4.9] on the simply-connected domain Ω shows the existence $a \in H^{\alpha}(\Omega)$ with $\nabla^{\perp} a = \operatorname{Div}\eta$ and $\|a\|_{H^{\alpha}(\Omega)} \lesssim \|\eta\|_{H^{\alpha}(\Omega)}$. For $J = (0,-1;1,0)$, the rows of $\eta + aJ$ are divergence-free. Invoking the same result row-wise and taking symmetric parts yields a potential $q \in H^{1+\alpha}(\Omega;\mathbb{R}^2)$ satisfying

$$\nabla_{\mathbb{S}}^{\perp} q = \eta, \qquad \|q\|_{H^{1+\alpha}(\Omega)} \lesssim \|\eta\|_{H^{\alpha}(\Omega)}.$$

The field q is unique among the choices orthogonal to RT. Taking $\alpha = 0$ proves the claimed decomposition and the asserted H^1 bound. For the proof of uniqueness of the decomposition, it suffices to consider the zero decomposition $pI + \nabla_{\mathbb{S}}^{\perp} q = 0$. Taking the double divergence gives $\Delta p = 0$, and the unique solvability of the Dirichlet-Laplacian shows uniqueness of p. The uniqueness of q follows from orthogonality to RT. Finally, if $\tau \in H^{\alpha}(\Omega;\mathbb{S})$ with $0 \leq \alpha \leq 3/2+\varepsilon$, then

$$\|\eta\|_{H^{\alpha}(\Omega)} \lesssim \|\tau\|_{H^{\alpha}(\Omega)} + \|\operatorname{divDiv}\tau\|_{L^2(\Omega)}.$$

The preceding potential construction and uniqueness give (6.2) for the same q.

For the proof of the boundary regularity, let τ be H^s regular with $s > 1/2$. It was already shown that $q \in H^{1+\beta}(\Omega;\mathbb{R}^2)$ for $\beta = \min\{s, 1/2+\varepsilon\}$, thus $q|_{\partial\Omega} \in H^1(\partial\Omega;\mathbb{R}^2)$. If τ is piecewise polynomial, $\eta = \nabla_{\mathbb{S}}^{\perp} q = \tau - pI$ in the above construction satisfies $\eta|_K \in H^{3/2+\varepsilon}(K;\mathbb{S})$ on every element K. Taking derivatives in the components of the matrix $\eta = \nabla_{\mathbb{S}}^{\perp} q$ gives the (distributional) relations for any pair $j \neq k$ from $\{1,2\}$ as follows

$$\partial_{jj} q_j = (-1)^j (2\partial_j \eta_{jk} + \partial_k \eta_{kk}), \partial_{jk} q_j = (-1)^k \partial_j \eta_{jj}, \partial_{jj} q_k = (-1)^j \partial_j \eta_{kk}.$$

Thus $q|_K \in H^{5/2+\varepsilon}(K;\mathbb{R}^2)$. The embedding $H^{5/2+\varepsilon}(K) \hookrightarrow C^0(\overline{K})$ provides continuous representatives on each closed element. Since $q \in H^1(\Omega;\mathbb{R}^2)$, the traces of adjacent elements coincide almost everywhere on their common edge. By continuity, they coincide on the whole edge, including its endpoints. Propagating this equality through each vertex patch shows that the boundary edge traces have matching endpoint values, which, together with the edge-wise H^1 regularity, gives $H^1(\partial\Omega;\mathbb{R}^2)$. ∎

Remark 6.2. Since $\varepsilon = 1/2$ on convex domains and $s < 1$ on non-convex polygons [28, Chapter 3], we always have $s \leq 3/2 + \varepsilon$ for ε from Lemma 6.1.

For $g = (g_1, g_2) \in X$ and $\tau \in \Sigma^r + \Sigma_h$, let $\tau = pI + \nabla_{\mathbb{S}}^{\perp} q$ be the unique decomposition in Lemma 6.1. Since $p|_{\partial\Omega} = 0$, define

$$\langle g, \tau \rangle := (\partial_n p, g_1)_{L^2(\partial\Omega)} - (\partial_t q, t\,\partial_t g_1 + n\,g_2)_{L^2(\partial\Omega)}. \tag{6.3}$$

Here $\partial_t q$ denotes the tangential derivative of $q|_{\partial\Omega}$. The regularity of p and the last statement of Lemma 6.1 together with the linearity ensure that both terms are well defined for any $\tau \in \Sigma^r + \Sigma_h$. Moreover, the decomposition estimates and the trace theorem give $|\langle g, \tau \rangle| \lesssim \|g\|_X \|\tau\|_{\Sigma^r}$ for all $\tau \in \Sigma^r$, so g induces a continuous functional on Σ^r.

Remark 6.3. The boundary functional (6.3) agrees with the trace defined via Green's identity as

$$\langle g, \tau \rangle = (\operatorname{divDiv} \tau, G)_{L^2(\Omega)} - (\tau, \nabla^2 G)_{L^2(\Omega)},$$

whenever g admits an $H^2(\Omega)$ lifting G. Indeed, assume that $g_1 = G|_{\partial\Omega}$ and $g_2 = \partial_n G$ for some $G \in H^2(\Omega)$. Since $\Delta p = \operatorname{divDiv} \tau$ and $p|_{\partial\Omega} = 0$, Green's formula gives $(\Delta p, G)_{L^2(\Omega)} - (p, \Delta G)_{L^2(\Omega)} = (\partial_n p, g_1)_{L^2(\partial\Omega)}$. Since the rows of $\nabla^{\perp} q$ are divergence-free with normal trace $\partial_t q$, it holds that $(\nabla_{\mathbb{S}}^{\perp} q, \nabla^2 G)_{L^2(\Omega)} = (\partial_t q, \nabla G)_{L^2(\partial\Omega)} = (\partial_t q, t\,\partial_t g_1 + n\,g_2)_{L^2(\partial\Omega)}$. This implies that the definition in (6.3) is consistent with the usual divDiv traces for regular boundary data.

Remark 6.4. For $\tau_h \in \Sigma_h$, the boundary functional (6.3) coincides with the boundary functional used in the discrete formulation of [24]. Indeed, this follows from Remark 6.3, the element-wise Green formula and density and continuity of both boundary functionals with respect to the norm of X. This result provides an extension of its boundary action to the test space required by the transposition formulation.

We next estimate the discrete boundary functional. Let $\tau_h = pI + \nabla_{\mathbb{S}}^{\perp} q$ be the unique decomposition of $\tau_h \in \Sigma_h$ from Lemma 6.1. Since τ_h is piecewise polynomial and thus H^{α}-regular for any $\alpha < 1/2$, the q component has global $H^{1+\alpha}$ regularity. The triangle inequality gives

$$|\langle g, \tau_h \rangle| \leq |(\partial_n p, g_1)_{L^2(\partial\Omega)}| + |(\partial_t q,\, t\,\partial_t g_1)_{L^2(\partial\Omega)}| + |(\partial_t q,\, n\,g_2)_{L^2(\partial\Omega)}|. \tag{6.4}$$

The first term can be bounded by $|(\partial_n p, g_1)_{L^2(\partial\Omega)}| \lesssim \|\operatorname{divDiv} \tau_h\|_{L^2(\Omega)} \|g_1\|_{L^2(\partial\Omega)}$. For the second term with $t_1 = 1$, it follows from the scaled trace inequality that

$$\begin{aligned} |(\partial_t q, t\,\partial_t g_1)_{L^2(\partial\Omega)}| &\leq \|\partial_t q\|_{L^2(\partial\Omega)} \|\partial_t g_1\|_{L^2(\partial\Omega)} \\ &\lesssim \|g_1\|_{H^1(\partial\Omega)} (h^{-1/2} \|q\|_{H^1(\Omega)} + h^{1/2} \|\nabla_h^2 q\|_{L^2(\Omega)}). \end{aligned} \tag{6.5}$$

The local differential identities in the proof of Lemma 6.1 and the polynomial inverse inequality imply $\|\nabla_h^2 q\|_{L^2(\Omega)} \lesssim \|\nabla_h(\tau_h - pI)\|_{L^2(\Omega)} \lesssim h^{-1} \|\tau_h\|_{L^2(\Omega)} + \|p\|_{H^1(\Omega)}$. This, (6.5) and (6.2) give $|(\partial_t q, t\,\partial_t g_1)_{L^2(\partial\Omega)}| \lesssim h^{-1/2} \|g_1\|_{H^1(\partial\Omega)} \|\tau_h\|_{\Sigma}$. For the second term of (6.4) with $1 < t_1 < 3/2$, it follows from the trace theorem that

$$\begin{aligned} |(\partial_t q, t\,\partial_t g_1)_{L^2(\partial\Omega)}| &\lesssim \|\partial_t q\|_{H^{1-t_1}(\partial\Omega)} \|\partial_t g_1\|_{H^{t_1-1}(\partial\Omega)} \lesssim \|q\|_{H^{5/2-t_1}(\Omega)} \|g_1\|_{H^{t_1}(\partial\Omega)} \\ &\lesssim (\|\tau_h\|_{H^{3/2-t_1}(\Omega)} + \|\operatorname{divDiv} \tau_h\|_{L^2(\Omega)}) \|g_1\|_{H^{t_1}(\partial\Omega)} \lesssim h^{t_1 - 3/2} \|g_1\|_{H^{t_1}(\partial\Omega)} \|\tau_h\|_{\Sigma}, \end{aligned}$$

where the last two estimates follow by (6.2) and the inverse estimate for τ_h. Similar considerations for the third term of (6.4) give

$$\begin{aligned} |(\partial_t q,\, n\,g_2)_{L^2(\partial\Omega)}| &\lesssim h^{-1/2} \|g_2\|_{L^2(\partial\Omega)} \|\tau_h\|_{\Sigma} && \text{for } t_2 = 0, \\ |(\partial_t q,\, n\,g_2)_{L^2(\partial\Omega)}| &\lesssim h^{t_2 - 1/2} \|g_2\|_{H^{t_2}(\partial\Omega)} \|\tau_h\|_{\Sigma} && \text{for } 0 < t_2 < 1/2. \end{aligned}$$

Substituting these estimates into (6.4) gives the discrete stability

$$\|\sigma_h\|_\Sigma + \|u_h\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} + h^{t_1-3/2}\|g_1\|_{H^{t_1}(\partial\Omega)} + h^{t_2-1/2}\|g_2\|_{H^{t_2}(\partial\Omega)}.$$

Let $P_h : L^2(\Omega) \to U_h$ denote the L^2-projection, which satisfies

$$\|w - P_h w\|_{L^2(\Omega)} \le Ch^2\|w\|_{U^r} \quad \text{for all } w \in U^r$$

so that Assumption 4 holds with $\rho_U = h^2$. Let Π_h^{FH} denote the Führer–Heuer interpolant in [24, Section 3.3]. Let I_h^2 be the continuous quadratic Lagrange interpolant for vector fields. For $\phi \in \Sigma^r$, let $\phi = pI + \nabla_{\mathbb{S}}^\perp q$ be its unique decomposition and define

$$\Pi_h\phi = \Pi_h^{\mathrm{FH}}(pI) + \nabla_{\mathbb{S}}^\perp(I_h^2 q), \quad \delta_p = pI - \Pi_h^{\mathrm{FH}}(pI), \quad \delta_q = q - I_h^2 q. \tag{6.6}$$

By [24, Proposition 10], it holds that

$$\operatorname{divDiv}\Pi_h^{\mathrm{FH}}(pI) = P_h\Delta p, \quad \|pI - \Pi_h^{\mathrm{FH}}(pI)\|_{L^2(\Omega)} \lesssim h^{3/2+\varepsilon}\|p\|_{H^{3/2+\varepsilon}(\Omega)}. \tag{6.7}$$

The nodal values of q are well defined because $q \in H^{1+s}(\Omega;\mathbb{R}^2)$, with $s > 1/2$. Since $P_2(K;\mathbb{R}^2) \subset RT_2(K)$, $\nabla_{\mathbb{S}}^\perp(I_h^2 q)|_K$ belongs to the local shape-function space. Moreover, $I_h^2 q \in H^1(\Omega;\mathbb{R}^2)$, so its symmetric curl has vanishing distributional double divergence. Thus $\Pi_h\phi \in \Sigma_h$, and $\operatorname{divDiv}\Pi_h\phi = P_h\Delta p = P_h \operatorname{divDiv}\phi$. Assumption 5(i) holds. The approximation estimates for the two components give

$$\|\phi - \Pi_h\phi\|_{L^2(\Omega)} \lesssim h^{3/2+\varepsilon}\|p\|_{H^{3/2+\varepsilon}(\Omega)} + h^s\|q\|_{H^{1+s}(\Omega)} \lesssim h^s\|\phi\|_{\Sigma^r},$$

where we used Remark 6.2. Thus $\Pi_h : \Sigma^r \to \Sigma_h$ is a continuous linear commuting approximation operator. This gives Assumption 5(ii) with $\rho_\Sigma = h^s$. For $g \in X$, since $p|_{\partial\Omega} = 0$, preservation of the normal-normal edge moments and $(\Pi_h^{\mathrm{FH}}(pI))_{nn}|_e \in P_1(e)$ imply $(\Pi_h^{\mathrm{FH}}(pI))_{nn} = 0$ on every boundary edge. Consequently, $\langle(0, g_2), \delta_p\rangle = 0$ and

$$\langle g, \phi - \Pi_h\phi\rangle = \langle g, \delta_p\rangle + \langle g, \nabla_{\mathbb{S}}^\perp\delta_q\rangle = \langle \left(\begin{smallmatrix} g_1 \\ 0 \end{smallmatrix}\right), \delta_p\rangle + \langle \left(\begin{smallmatrix} g_1 \\ 0 \end{smallmatrix}\right), \nabla_{\mathbb{S}}^\perp\delta_q\rangle + \langle \left(\begin{smallmatrix} 0 \\ g_2 \end{smallmatrix}\right), \nabla_{\mathbb{S}}^\perp\delta_q\rangle.$$

By Remark 6.4, the boundary functional defined in (6.3) agrees on Σ_h with the boundary action of [24]. The interpolant Π_h^{FH} preserves the $P_1(e)$-moments of the normal-normal and effective-shear traces as well as the corner-force degrees of freedom [24, eqs. (5)–(6)]. For pI, the normal-normal trace equals p, the effective-shear trace equals $\partial_n p$, and all corner-force contributions vanish. Since $p|_{\partial\Omega} = 0$, it follows that

$$\langle \left(\begin{smallmatrix} g_1 \\ 0 \end{smallmatrix}\right), \delta_p\rangle = \sum_{e\in\mathcal{E}_h^\partial} \big((1-Q_e^1)\partial_n p,\, g_1\big)_{L^2(e)} \le \sum_{e\in\mathcal{E}_h^\partial} \|(1-Q_e^1)\partial_n p\|_{L^2(e)}\|(1-Q_e^1)g_1\|_{L^2(e)}.$$

For $e \in \mathcal{E}(K)$, subtracting an affine approximation of p, applying the scaled trace inequality, and using the approximation property of Q_e^1 yield $\|(1-Q_e^1)\partial_n p\|_{L^2(e)} \lesssim h_K^\varepsilon |p|_{H^{3/2+\varepsilon}(K)}$. Indeed, the affine approximation has a constant normal derivative, while its restriction to e belongs to $P_1(e)$; both are annihilated by $1 - Q_e^1$. This, the edge projection estimates for the data and summing over boundary edges gives

$$|\langle(g_1, 0), \delta_p\rangle| \lesssim h^{t_1+\varepsilon}\|g_1\|_{H^{t_1}(\partial\Omega)}\|p\|_{H^{3/2+\varepsilon}(\Omega)}. \tag{6.8}$$

By the global boundary identity,

$$\langle(g_1, 0), \nabla_{\mathbb{S}}^\perp\delta_q\rangle = -(\partial_t\delta_q, t\,\partial_t g_1)_{L^2(\partial\Omega)}.$$

For $t_1 = 1$, the Cauchy–Schwarz inequality and the derivative trace estimate for quadratic Lagrange approximation directly give

$$|\langle(g_1, 0), \nabla_{\mathbb{S}}^\perp\delta_q\rangle| \lesssim \|\partial_t\delta_q\|_{L^2(\partial\Omega)}\|t\,\partial_t g_1\|_{L^2(\partial\Omega)} \lesssim h^{s-1/2}\|g_1\|_{H^1(\partial\Omega)}\|q\|_{H^{1+s}(\Omega)}.$$

For $1 < t_1 < 3/2$, note that $I_h^2 q$ preserves the values at both endpoints of every mesh edge, then $\int_e \partial_t\delta_q\, ds = 0$ for every $e \in \mathcal{E}_h^\partial$. This and t is constant on e give

	Unit square		L-shaped domain	
$h_\ell/\sqrt{2}$	$\|u-u_h\|_{L^2(\Omega)}$	order	$\|u-u_h\|_{L^2(\Omega)}$	order
2^{-1}	5.3871e−3	–	1.0866e−2	–
2^{-2}	1.8247e−3	1.5618	7.2860e−3	0.5766
2^{-3}	5.9664e−4	1.6127	6.2021e−3	0.2324
2^{-4}	1.9167e−4	1.6382	5.4977e−3	0.1739
2^{-5}	6.1105e−5	1.6493	4.8311e−3	0.1865
2^{-6}	1.9393e−5	1.6557	4.2090e−3	0.1989
2^{-7}	6.1370e−6	1.6600	3.6502e−3	0.2055

TABLE 4. L^2 errors and observed convergence orders for the lowest-order Führer–Heuer element approximation of the two-dimensional biharmonic problem with the boundary data $g_1 \in H^{7/6-\nu}$ and $g_2 \in H^{1/6-\nu}$.

$(\partial_t\delta_q, t\,\partial_t g_1)_{L^2(e)} = (\partial_t\delta_q,\, t(\partial_t g_1 - Q_e^0\partial_t g_1))_{L^2(e)}$. Therefore, the Cauchy–Schwarz inequality and the fractional Poincaré estimate on each edge yield

$$|\langle (g_1,0), \nabla_{\mathbb{S}}^{\perp}\delta_q\rangle| \lesssim h^{s+t_1-3/2}\|g_1\|_{H^{t_1}(\partial\Omega)}\|q\|_{H^{1+s}(\Omega)}. \tag{6.9}$$

The discussion of $t_1 = 1$ is covered also by (6.9). Following similar procedures as above, we can verify the estimate of the term

$$|\langle (0,g_2), \nabla_{\mathbb{S}}^{\perp}\delta_q\rangle| = |(\partial_t\delta_q, ng_2)_{L^2(\partial\Omega)}| \lesssim h^{s+t_2-1/2}\|g_2\|_{H^{t_2}(\partial\Omega)}\|q\|_{H^{1+s}(\Omega)}.$$

Thus, for every $\phi \in \Sigma^r$ and $g \in X$,

$$|\langle g, \phi - \Pi_h\phi\rangle| \lesssim \Big(h^{s+t_1-3/2}\|g_1\|_{H^{t_1}(\partial\Omega)} + h^{s+t_2-1/2}\|g_2\|_{H^{t_2}(\partial\Omega)}\Big)\|\phi\|_{\Sigma^r}.$$

Assumption 5(iii) holds with $\rho_X = h^{\min\{s+t_1-3/2, s+t_2-1/2\}}$. Since $\Sigma_h \subset \Sigma$, $\rho_{\mathrm{nc}} = 0$.

Corollary 6.5. Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected polygonal Lipschitz domain, and let $1/2 < s \le 2$ be an admissible regularity index for the homogeneous clamped biharmonic problem. For $f \in L^2(\Omega)$ and $g = (g_1, g_2) \in X = H^{t_1}(\partial\Omega) \times H^{t_2}(\partial\Omega)$ with $1 \le t_1 < 3/2$ and $0 \le t_2 < 1/2$. Let u be the transposition solution of the biharmonic equation (6.1) and let $(\sigma_h, u_h) \in \Sigma_h \times U_h$ solve the mixed method. Then it holds that

$$\|u-u_h\|_{L^2(\Omega)} \lesssim h^s\|f\|_{L^2(\Omega)} + h^{s+t_1-3/2}\|g_1\|_{H^{t_1}(\partial\Omega)} + h^{s+t_2-1/2}\|g_2\|_{H^{t_2}(\partial\Omega)}.$$

Example 6.6. We consider the biharmonic problem on the square and L-shaped domains from Example 3.3 and the exact solution

$$u(r,\theta) = r^{1+\alpha}\sin\big((1+\alpha)\theta\big), \qquad \alpha = -1/3.$$

It satisfies $f = \Delta^2 u = 0$. The boundary terms satisfy

$$g_1 \in H^{7/6-\nu}(\partial\Omega), \qquad g_2 \in H^{1/6-\nu}(\partial\Omega) \quad \text{for all } \nu > 0.$$

On the square domain, the regularity-shift index of the adjoint clamped-plate problem $s_S = 2$, and hence $\|u-u_h\|_{L^2(\Omega)}$ behaves like $h^{5/3-\nu}$. For the standard L-shaped domain, the first clamped-plate singular exponent gives $s_L = 0.544\ldots$, so that the predicted rate is $s_L + \alpha = 0.211\ldots$. Table 4 shows L^2 errors and convergence rates of the lowest-order element, which coincide with the theoretical results of Corollary 6.5. □

7. Application to Maxwell equation

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected Lipschitz polyhedron with connected boundary $\partial\Omega$. We consider the curl-curl problem

$$\operatorname{curl}\operatorname{curl} u = f, \quad \operatorname{div} u = 0 \text{ in } \Omega, \quad n \times u = g \text{ on } \partial\Omega.$$

We assume the compatibility conditions $\operatorname{div} f = 0$ in Ω and $g \cdot n = 0$ on $\partial\Omega$. The mixed formulation is based on the auxiliary variable $\sigma = \operatorname{curl} u$ and the spaces

$$\Sigma = H(\operatorname{curl}, \Omega; \mathbb{R}^3) := \{\tau \in L^2(\Omega; \mathbb{R}^3) : \operatorname{curl}\tau \in L^2(\Omega; \mathbb{R}^3)\},$$
$$U = H(\operatorname{div}^0, \Omega; \mathbb{R}^3) := \{v \in H(\operatorname{div}, \Omega; \mathbb{R}^3) : \operatorname{div} v = 0 \text{ in } \Omega\},$$

equipped with the norms $\|\cdot\|_\Sigma = \|\cdot\|_{L^2(\Omega)} + \|\operatorname{curl}\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_U = \|\cdot\|_{L^2(\Omega)}$, respectively. We consider the bilinear forms

$$a(\sigma, \tau) = (\sigma, \tau)_{L^2(\Omega)}, \quad b(\tau, v) = -(\operatorname{curl}\tau, v)_{L^2(\Omega)} \text{ for all } \sigma, \tau \in \Sigma, v \in U.$$

As before, the form a extends to $H = L^2(\Omega; \mathbb{R}^3)$ equipped with the L^2 norm. Assumption 1 follows from the mixed formulation of the curl-curl problem [34]. Let $s \in (1/2, 1]$ be an admissible elliptic shift exponent so that the solution to the homogeneous Dirichlet problem of the curl-curl equation is $H^s(\Omega)$ regular [18]. By $H_0(\operatorname{curl}, \Omega; \mathbb{R}^3)$ we denote the closure of smooth and compactly supported vector fields with respect to the norm $\|\cdot\|_\Sigma$. Let $U^r = H^s(\Omega; \mathbb{R}^3) \cap H_0(\operatorname{curl}, \Omega; \mathbb{R}^3) \cap U$ with the norm $\|\cdot\|_{U^r} = \|\cdot\|_{H^s(\Omega)} + \|\operatorname{curl}\cdot\|_{L^2(\Omega)}$ and

$$\Sigma^r = \Sigma \cap H^s(\Omega; \mathbb{R}^3) \text{ with the norm } \|\cdot\|_{\Sigma^r} = \|\cdot\|_{H^s(\Omega)} + \|\operatorname{curl}\cdot\|_{L^2(\Omega)}.$$

By [18], for any $z \in U$, the solution to $T(\phi, w) = (0, z)$ is unique and satisfies

$$\|w\|_{H^s(\Omega)} + \|\phi\|_{H^s(\Omega)} \lesssim \|z\|_{L^2(\Omega)}.$$

This verifies Assumption 2.

Let $\mathcal{T}_h$ be a tetrahedral triangulation of Ω as in prior sections. Given $K \in \mathcal{T}_h$, we define the usual Nédélec space as $N_0(K) := \{\tau : K \to \mathbb{R}^3 : \tau(x) = \alpha + \beta \times x \text{ for } \alpha, \beta \in \mathbb{R}^3\}$. We consider the conforming lowest-order discrete de Rham pair in [12]

$$\Sigma_h = \{\tau \in \Sigma : \tau|_K \in N_0(K) \text{ for all } K \in \mathcal{T}_h\}, \qquad U_h = \operatorname{curl}\Sigma_h.$$

By exactness of the discrete de Rham complex, U_h equals the divergence-free lowest-order Raviart–Thomas space. The corresponding discrete mixed problem is uniformly stable [34], which gives Assumption 3. For any $\tau \in \Sigma$, let $\gamma_t\tau := (n \times (\tau \times n))|_{\partial\Omega}$ denote its tangential component. For $0 \le t < 1/2$, define $X = \{g \in H^t(\partial\Omega; \mathbb{R}^3) : g \cdot n = 0 \text{ on } \partial\Omega\}$ with $\|g\|_X := \|g\|_{H^t(\partial\Omega)}$.

Remark 7.1. We consider the Maxwell problem with H^t boundary data, but unlike in the other PDE examples of this paper, this does not describe a superspace of the trace space of Σ, which is described in [14]. Still, boundary data in H^t can admit functions that need not belong to the the standard Maxwell trace space.

For $\tau_h \in \Sigma_h$, define $\langle g, \tau_h\rangle = (g, \gamma_t\tau_h)_{L^2(\partial\Omega)}$. For $t = 0$, the trace and inverse inequalities show

$$|\langle g, \tau_h\rangle| \le \|g\|_X \|\tau_h\|_{L^2(\partial\Omega)} \lesssim h^{-1/2}\|g\|_X\|\tau_h\|_\Sigma. \tag{7.1}$$

For $0 < t < 1/2$, it holds that $g \times n \in H^t(\partial\Omega; \mathbb{R}^3)$ and $\|g \times n\|_{H^t(\partial\Omega)} \lesssim \|g\|_X$, see [29, Corollary 1.4.4.5]. By [29, Theorem 1.5.1.2], there exists a lifting $G \in H^{1/2+t}(\Omega; \mathbb{R}^3)$ such that $G|_{\partial\Omega} = g \times n$ and $\|G\|_{H^{1/2+t}(\Omega)} \lesssim \|g\|_X$. Since $g \cdot n = 0$, we have $n \times G = n \times (g \times n) = g$ and the generalized Green formula

$$\langle g, \tau_h\rangle = \langle \operatorname{curl} G, \tau_h\rangle_\Omega - (G, \operatorname{curl}\tau_h)_{L^2(\Omega)}. \tag{7.2}$$

The first term pairs $H^{t-1/2}(\Omega)$ with $H_0^{1/2-t}(\Omega) = H^{1/2-t}(\Omega)$. This and the inverse estimate give $|\langle \operatorname{curl} G, \tau_h\rangle_\Omega| \le \|\operatorname{curl} G\|_{H^{t-1/2}(\Omega)} \|\tau_h\|_{H^{1/2-t}(\Omega)} \lesssim h^{t-1/2}\|g\|_X \|\tau_h\|_\Sigma$. The second term $|(G, \operatorname{curl}\tau_h)_{L^2(\Omega)}| \le \|G\|_{L^2(\Omega)} \|\operatorname{curl}\tau_h\|_{L^2(\Omega)} \lesssim \|g\|_X\|\tau_h\|_\Sigma$. Substituting these two estimates into (7.2) and combining with (7.1) results in

$$\|\sigma_h\|_\Sigma + \|u_h\|_U \lesssim \|f\|_{L^2} + h^{t-1/2}\|g\|_X. \tag{7.3}$$

Let $P_h : U^r \to U_h$ denote the lowest-order Raviart–Thomas interpolant [11]. Since $\operatorname{div} w = 0$ for $w \in U^r$, the commuting property implies $\operatorname{div} P_h w = 0$, and therefore $P_h w \in U_h$. Hence, $\|w - P_h w\|_{L^2(\Omega)} \lesssim h^s \|w\|_{H^s(\Omega)}$ and Assumption 4 holds with $\rho_U = h^s$. Let $\Pi_h : \Sigma^r \to \Sigma_h$ denote the canonical Nédélec interpolant [11]. It does not, in general, satisfy the Fortin property $b(\Pi_h\phi, v_h) = b(\phi, v_h)$ for all $\phi \in \Sigma^r$ and $v_h \in U_h$, since the Raviart–Thomas interpolant is not the L^2-orthogonal projection onto U_h. However, if $T(\phi, w) = (0, z_h)$ for some $z_h \in U_h$, then $\operatorname{curl}\phi \in U_h$ and $P_h \operatorname{curl}\phi = \operatorname{curl}\phi$. This implies

$$b(\Pi_h\phi, v_h) = -(\operatorname{curl}\Pi_h\phi, v_h)_{L^2(\Omega)} = -(P_h \operatorname{curl}\phi, v_h)_{L^2(\Omega)} = b(\phi, v_h)$$

for all $v_h \in U_h$. This proves Assumption 5(i). By [34, 11],

$$\|\phi - \Pi_h\phi\|_{L^2(\Omega)} \lesssim h^s\|\phi\|_{H^s(\Omega)} + h\|\operatorname{curl}\phi\|_{L^2(\Omega)}. \tag{7.4}$$

Using the regularity of the dual solution, we obtain

$$\|\phi - \Pi_h\phi\|_{L^2(\Omega)} \lesssim h^s\|z_h\|_{L^2(\Omega)}.$$

Assumption 5(ii) holds with $\rho_\Sigma = h^s$. Let $\delta_h := \phi - \Pi_h\phi$. Since $\operatorname{curl}\phi \in U_h$, it satisfies $\operatorname{curl}\delta_h = 0$. We next estimate the boundary term $|\langle g, \delta_h\rangle|$. Since δ_h is not a discrete function, we introduce a further approximation step. Let q_h be a continuous piecewise polynomial quasi-interpolant of ϕ satisfying the standard approximation properties. For $t = 0$, the triangle inequality, the boundary approximation estimate, and the inverse trace inequality applied to $q_h - \Pi_h\phi$ give

$$\begin{aligned}\|\delta_h\|_{L^2(\partial\Omega)} &\lesssim \|\phi - q_h\|_{L^2(\partial\Omega)} + h^{-1/2}\|q_h - \Pi_h\phi\|_{L^2(\Omega)}\\ &\lesssim h^{s-1/2}\|\phi\|_{H^s(\Omega)} + h^{-1/2}\|\delta_h\|_{L^2(\Omega)}.\end{aligned}$$

Together with (7.4) and the inverse estimate for the discrete function $\operatorname{curl}\phi$, this yields

$$|\langle g, \delta_h\rangle| \le \|g\|_X\|\delta_h\|_{L^2(\partial\Omega)} \lesssim h^{s-1/2}\|g\|_X\|\phi\|_{H^s(\Omega)}. \tag{7.5}$$

Let now $0 < t < 1/2$, and let G be the lifting introduced above. By (7.2) and $\operatorname{curl}\delta_h = 0$, the generalized Green formula simplifies to $\langle g, \delta_h\rangle = \langle \operatorname{curl} G, \delta_h\rangle_\Omega$. Furthermore,

$$\begin{aligned}\|\delta_h\|_{H^{1/2-t}(\Omega)} &\lesssim \|\phi - q_h\|_{H^{1/2-t}(\Omega)} + h^{t-1/2}\|q_h - \Pi_h\phi\|_{L^2(\Omega)}\\ &\lesssim h^{s+t-1/2}\|\phi\|_{H^s(\Omega)} + h^{t-1/2}\|\delta_h\|_{L^2(\Omega)}.\end{aligned}$$

This, combined with (7.4), gives the estimate

$$|\langle g, \delta_h\rangle| = |\langle \operatorname{curl} G, \delta_h\rangle_\Omega| \le \|\operatorname{curl} G\|_{H^{t-\frac12}(\Omega)}\|\delta_h\|_{H^{\frac12-t}(\Omega)} \lesssim h^{s+t-\frac12}\|g\|_X\|\phi\|_{H^s(\Omega)}.$$

This and (7.5) verify Assumption 5(iii) with $\rho_X = h^{s+t-1/2}$. Since $\Sigma_h \subset \Sigma$, Remark 2.5 applies and $\rho_{\mathrm{nc}} = 0$.

Corollary 7.2. Let $\Omega \subseteq \mathbb{R}^3$ be an open, bounded simply-connected Lipschitz polyhedron with connected boundary and let $1/2 < s \le 1$ be an admissible regularity index for the homogenuous Dirichlet curl-curl problem. Given $f \in L^2(\Omega;\mathbb{R}^3)$ and $g \in H^t(\partial\Omega;\mathbb{R}^3)$ with $0 \le t < 1/2$ satisfying the compatibility conditions $\operatorname{div} f = 0$

	Unit cube		L-shaped prism	
$h_\ell/\sqrt{3}$	$\|u-u_h\|_{L^2(\Omega)}$	order	$\|u-u_h\|_{L^2(\Omega)}$	order
2^{-1}	8.6655e−2	–	2.3272e−1	–
2^{-2}	6.3194e−2	0.4555	1.7818e−1	0.3853
2^{-3}	4.3248e−2	0.5472	1.3295e−1	0.4224
2^{-4}	2.8424e−2	0.6055	9.8980e−2	0.4257
2^{-5}	1.8306e−2	0.6348	7.4432e−2	0.4112
2^{-6}	1.1670e−2	0.6495	5.6699e−2	0.3926

TABLE 5. L^2 errors and observed convergence orders for the lowest-order Nédélec approximation of the three-dimensional Maxwell problem with boundary data $g \in H^{1/6-\nu}(\partial\Omega)$.

in Ω and $g\cdot n = 0$ on $\partial\Omega$, let u be the transposition solution of the curl-curl problem and let (σ_h, u_h) solve the discrete mixed problem. Then it holds that

$$\|u-u_h\|_{L^2(\Omega)} \lesssim \inf_{v_h\in U_h} \|u-v_h\|_{L^2(\Omega)} + h^s\|f\|_{L^2(\Omega)} + h^{s+t-1/2}\|g\|_X.$$

Remark 7.3. If, in addition, $u \in H^r(\Omega;\mathbb{R}^3)$ with $r \geq 0$, then it can be shown [23] that the best-approximation term on the right-hand side of the error estimate in Corollary 7.2 is controlled by $h^{\min\{r,1\}}\|u\|_{H^r(\Omega)}$. Hence

$$\|u-u_h\|_{L^2(\Omega)} \lesssim h^{\min\{r,1\}}\|u\|_{H^r(\Omega)} + h^s\|f\|_{L^2(\Omega)} + h^{s+t-1/2}\|g\|_X.$$

Example 7.4. We consider the three-dimensional Maxwell problem on the unit cube Ω_C and the L-shaped prism Ω_P from Example 3.3. The exact solution is $u(x,y,z) = \big(0,0,r^\alpha\sin(\alpha\theta)\big)^T$ with $\alpha = -1/3$ and vanishing source term $f = 0$. The tangential boundary datum $g = n\times u \in H^{1/6-\nu}(\partial\Omega;\mathbb{R}^3)$ is imposed weakly through the natural duality pairing in the transposition formulation.

Table 5 reports the L^2 errors and the observed convergence orders for the lowest-order Nédélec edge element approximation. Since $u \in H^{3/2-\nu}(\Omega;\mathbb{R}^3)$, Remark 7.3 shows the first term of the error estimate gives $h^{3/2-\nu}\|u\|_{H^{3/2-\nu}(\Omega)}$. On the unit cube, the observed convergence rate is close to the theoretical prediction $s+t-1/2 = 2/3-\nu$ (with shift index $s=1$ and boundary exponent $t = 1/6-\nu$). For the L-shaped prism, the edge singularity limits the regularity index to $s_P = 2/3-\nu$, yielding an expected convergence order of $s_P + t - 1/2 = 1/3-\nu$ for any ν. The observed convergence rates are close to this prediction. □

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