On the cohomology of the sporadic simple group J_4

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Introduction

In this paper we calculate part of the integral cohomology ring of the sporadic simple group J_4: this group has order $2^{21}.3^3.5.7.11^3.23.29.31.37.43$. More precisely, we obtain all of the cohomology ring except for the 2-primary part. As the cohomology has already been written down [**9**] at the primes which divide the group order only once, we concentrate here on the primes 3 and 11. In both of these cases the Sylow p-subgroups are extraspecial of order p^3 and exponent p. We use the method which identifies the p-primary cohomology with the ring of *stable* classes in the cohomology of a Sylow p-subgroup. The stable classes are all invariant under the action of the Sylow p-normaliser; and some time is spent finding invariant classes in the cohomology ring of p^{1+2}_+, the extraspecial group. Section 2 studies the prime 11: the invariant classes are the stable classes, because the Sylow 11-subgroups have the Trivial Intersection (T.I.) property. In Section 3 we study the prime 3, and see that all conditions for invariant classes to be stable reduce to one condition.

In Section 4 we obtain the corresponding parts of the Chern subring of J_4. We make much use of generalised characters of the Sylow subgroups that are constant on the conjugacy classes of J_4. (Strictly, on the intersections of p^{1+2}_+ with the conjugacy classes of J_4.)

1 Preliminaries

In order to calculate the integral cohomology ring of a finite group G, it is enough to calculate the localisation of this ring at each prime p that divides the order of G. After localising at p, restriction from G to any Sylow p-subgroup P is an injection, since it is split up to a unit by corestriction. Theorem 1 characterises the image of this restriction map. Recall that a class x in the cohomology of P is *stable* if for all g in G the restriction of x to the intersection $P \cap P^g$ is equal to the restriction of g^*x from P^g to $P \cap P^g$, where g^* is induced by conjugation by g.

Theorem 1 *([**2**] XII 10.1) With the above notation, restriction from G to P identifies $\mathrm{H}^*(G, \mathbf{Z})_{(p)}$ with the ring of stable classes in $\mathrm{H}^*(P, \mathbf{Z})_{(p)}$.* ∎

If g normalises P, then g acts on the cohomology of P. The class x is *invariant* if it is fixed under this action of the Sylow p-normaliser. The ring $\mathrm{H}^{\mathrm{inv}}$ of invariant classes clearly contains the stable classes, and is simpler to calculate. In each case studied here, we start by finding the invariant classes and then determine which of them are stable.

The group $\mathbf{J_4}$ The group J_4 has order $2^{21}.3^3.5.7.11^3.23.29.31.37.43$, and is the fourth largest sporadic simple group. It was discovered by Janko [**4**] and proved to exist by Norton, Parker, Benson, Conway and Thackray [**8**], who constructed a representation in $GL_{112}(\mathbf{F}_2)$. Many properties of J_4 are listed in the Atlas [**3**].

For the primes which divide the order of J_4 only once, the Sylow p-subgroups are cyclic and calculating the cohomology is straightforward:

Theorem 2 *[**9**] Let p be a prime that divides the order of J_4 exactly once. Then $\mathrm{H}^*(J_4, \mathbf{Z})_{(p)}$ is generated by α_p, with degree as shown below:*

p	5	7	23	29	31	37	43
$\deg \alpha_p$	8	6	44	56	20	24	28
χ	χ_2	χ_2	χ_2	χ_2	χ_{56}	χ_{53}	χ_{46}

.

*The only relation is $p\alpha_p = 0$. The generator α_p is a Chern class of a representation of J_4 which affords the irreducible character χ given above. We have used Atlas notation [**3**], and chosen characters of lowest degree.* ∎

The primes that divide the order more than once are 2, 3 and 11. Since the cohomology of the Sylow 2-subgroup is not yet known, there are two primes, 3 and 11, where the calculations are both interesting and feasible. We now list the properties of J_4 that we use. We are interested in Sylow p-subgroups, their normalisers, and the ways in which Sylow p-subgroups intersect with each other.

Theorem 3 *([**3**], [**5**]) The Sylow 11-subgroups of J_4 are isomorphic to 11^{1+2}_+. Distinct Sylow 11-subgroups have trivial intersection. The centraliser of a Sylow 11-subgroup is its centre, and the normaliser is $11^{1+2}_+{:}\,(5 \times 2.S_4)$.*

There are two conjugacy classes of elements of order 11 in J_4. The central elements in 11^{1+2}_+ are in class $11A$, and the non-central elements are in class $11B$.

The Sylow 3-subgroups of J_4 are isomorphic to 3^{1+2}_+. The centraliser of a Sylow 3-subgroup is cyclic of order six, and the normaliser is $(2 \times 3^{1+2}_+{:}\,8){:}\,2$.

Since J_4 contains M_{12}, which in turn contains $3^2{:}\,2S_4$, there are Sylow 3-subgroups of J_4 that have different centres and intersect with order 3^2. All elements of order 3 in J_4 are in the conjugacy class $3A$. ∎

The extraspecial p-group p^{1+2}_+ The extraspecial p-group of order p^3 and exponent p is a Sylow p-subgroup of J_4 when p is 3 or 11. A presentation is

$$p^{1+2}_+ = \langle A, B, C\colon \quad A^p = B^p = C^p = [A,C] = [B,C] = 1, \quad [A,B] = C \rangle,$$

where $[X,Y]$ denotes $XYX^{-1}Y^{-1}$. The centre is cyclic of order p, generated by C.

The automorphisms of p^{1+2}_+ are those endomorphisms which send A to $A^{r'}B^{s'}C^{t'}$ and B to $A^r B^s C^t$, with $r's - rs'$ non-zero. The inner automorphisms are the automorphisms of the form $A \mapsto AC^{t'}$ and $B \mapsto BC^t$; so we can describe the outer automorphisms of p^{1+2}_+ thus:

Lemma 4 *The outer automorphism group of p_+^{1+2} is isomorphic to $GL_2\mathbf{F}_p$: the matrix $\left(\begin{smallmatrix} r' & r \\ s' & s \end{smallmatrix}\right)$ with determinant j corresponds to the class of the automorphism which sends A to $A^{r'}B^{s'}$, sends B to A^rB^s, and C to C^j.* ∎

(Note that the map we have given here is *not* a homomorphism from $GL_2\mathbf{F}_p$ to the automorphism group of p_+^{1+2}.)

We use this isomorphism to identify outer automorphisms of p_+^{1+2} with elements of $GL_2\mathbf{F}_p$. This group acts on the cohomology of p_+^{1+2}, because inner automorphisms of a group pass to the trivial automorphism in cohomology.

For a Sylow p-subgroup P of a group G, let Out_p denote the group of outer automorphisms of P which are induced by inner automorphisms of G; that is, the outer automorphisms that occur as conjugation by some element of the Sylow p-normaliser N_p.

We now give the cohomology of p_+^{1+2} and a subgroup: for proofs see [**7**].

Proposition 5 *Let p be an odd prime: then $\mathrm{H}^*(C_p^B \times C_p^C, \mathbf{Z})$ is generated by β and γ in degree 2, and χ in degree 3. All three generators have additive order p, and χ squares to zero. The automorphism of $C_p \times C_p$ which switches the two factors sends $\beta \leftrightarrow \gamma$ and $\chi \mapsto -\chi$.* ∎

Theorem 6 *([**7**], [**6**]) The cohomology ring $\mathrm{H}^*(p_+^{1+2}, \mathbf{Z})$ is generated by*

Generator	α_1, α_2	ν_1, ν_2	θ_j *for* $2 \leqslant j \leqslant p-2$	κ	ζ
Degree	2	3	$2j$	$2p-2$	$2p$
Additive order	p	p	p	p	p^2

.

The ν_i and the θ_j square to zero. The remaining relations (including some redundant ones) are:

$$\begin{array}{lll} \alpha_i\kappa = -\alpha_i^p & \alpha_1\nu_2 = \alpha_2\nu_1 & \alpha_1\alpha_2^p = \alpha_1^p\alpha_2 \\ \nu_i\kappa = -\alpha_i^{p-1}\nu_i & \kappa^2 = \alpha_1^{2p-2} - \alpha_1^{p-1}\alpha_2^{p-1} + \alpha_2^{2p-2} & \alpha_2^p\nu_1 = \alpha_1^p\nu_2 \end{array}$$

$$\alpha_i\theta_j = \nu_i\theta_j = \theta_k\theta_j = \kappa\theta_j = 0 \qquad \nu_1\nu_2 = \begin{cases} \lambda\theta_3 & \text{for some } \lambda \in \mathbf{F}_p^\times, \text{ if } p > 3 \\ 3\lambda\zeta & \text{for some } \lambda = \pm 1, \text{ if } p = 3. \end{cases}$$

Under the map of Lemma 4, the matrix $\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ with determinant j fixes κ, sends θ_k to $j^k\theta_k$, sends ζ to $j^p\zeta$ and sends

$$\alpha_i \mapsto a_{i1}\alpha_1 + a_{i2}\alpha_2 \qquad \nu_i \mapsto j(a_{i1}\nu_1 + a_{i2}\nu_2).$$

Under restriction to $C_p^B \times C_p^C$, the generators α_1, ν_1 and θ_j are sent to zero, and

$$\alpha_2 \mapsto \beta \qquad \nu_2 \mapsto \chi \qquad \kappa \mapsto -\beta^{p-1} \qquad \zeta \mapsto \gamma^p - \gamma\beta^{p-1}.$$ ∎

We derive two corollaries which help us to calculate inside this cohomology ring. The first looks at the additive structure, and the second relates the odd- and even-degree classes.

Corollary 7 *The ring* $\mathrm{H}^*(p_+^{1+2}, \mathbf{Z})$ *has the additive structure of the direct sum of three modules: the free* $\mathbf{Z}$*-module generated by* 1*, the free* $\mathbf{Z}/p^2$*-module on the* ζ^ℓ *for* $\ell > 0$*, and the* $\mathbf{F}_p$*-vector space with basis*

$$\begin{array}{lll} \alpha_1^i \alpha_2^j \zeta^\ell \text{ for } i > 0 \text{ and } j < p & \alpha_2^j \zeta^\ell \text{ for } j > 0 & \kappa\zeta^\ell \\ \alpha_1^i \alpha_2^j \nu_1 \zeta^\ell \text{ for } j < p & \alpha_2^j \nu_2 \zeta^\ell & \theta_k \zeta^\ell . \end{array}$$

Let V_1 *be the submodule spanned by the* $\alpha_1^i\alpha_2^j\zeta^\ell$ *and* $\alpha_2^j\zeta^\ell$*, let* V_2 *be the submodule spanned by the* $\alpha_1^i\alpha_2^j\nu_1\zeta^\ell$ *and* $\alpha_2^j\nu_2\zeta^\ell$*, and* V_3 *the submodule spanned by* 1*,* ζ^ℓ*,* $\theta_k\zeta^\ell$ *and* $\kappa\zeta^\ell$*. Then* $\mathrm{H}^*(p_+^{1+2}, \mathbf{Z})$ *is the direct sum* $V_1 \oplus V_2 \oplus V_3$*, and this decomposition is respected by the action of the automorphism group.*

Proof: Using the relations in Theorem 6, these classes are seen to span the cohomology ring. To see that they are linearly independent (in the obvious sense), consider the orders of the groups $\mathrm{H}^n(p_+^{1+2})$ as obtained in [**7**]. ∎

Corollary 8 *Let* $\mathcal{E}$ *be the ring of even-degree classes in* $\mathrm{H}^*(p_+^{1+2}, \mathbf{Z})$*. So* $\mathcal{E}$ *is* $V_1 \oplus V_3$*, and* V_2 *is the set of odd-degree classes. Also,* V_1 *is the* $\mathcal{E}$*-module generated by the* α_i*, and* V_2 *is the* $\mathcal{E}$*-module generated by the* ν_i*. Then the map* $\Phi\colon V_1 \to V_2$ *defined on the* generators *by* $\alpha_i \mapsto \nu_i$ *is a well-defined* $\mathcal{E}$*-module isomorphism.*

If x *and* y *are classes in* V_1*, then* $x(\Phi y) = \Phi(xy) = (\Phi x)y$*. If* ψ *is an automorphism of* p_+^{1+2} *that sends* C *to* C^j*, then* $\psi^*(\Phi x) = j\Phi(\psi^* x)$*.*

Proof: This is simple to check: for example, observe that $\Phi(\alpha_1\alpha_2)$ is well-defined. ∎

2 The 11-local cohomology

In this section we prove:

Theorem 9 *The ring* $\mathrm{H}^*(J_4, \mathbf{Z})_{(11)}$ *has generators as follows:*

Generator	α	β	μ	ν	ϕ_j *for* $2 \leqslant j \leqslant 9$	κ	η
Degree	56	148	35	127	$220 - 20j$	20	220
Additive order	11	11	11	11	11	11	11^2

.

The classes μ*,* ν *and* ϕ_j *square to zero, and the other relations are:*

$$\begin{array}{llll} \alpha\nu = \beta\mu & \alpha^3 = -\beta\kappa & \alpha^2\beta = \kappa^2\eta & \beta^2 = -\alpha\kappa\eta \\ \mu\nu = 0 & \alpha^2\mu = -\nu\kappa & & \beta\nu = -\mu\kappa\eta \end{array}$$
$$\alpha\phi_j = \beta\phi_j = \kappa\phi_j = \mu\phi_j = \nu\phi_j = \phi_i\phi_j = 0 \ .$$

Since J_4 is T.I. at 11, all the invariant classes are stable. Hence the 11-local cohomology of J_4 is the ring of invariant classes in the cohomology of 11_+^{1+2}; that is, the ring of classes fixed by the subgroup Out_{11} of $GL_2\mathbf{F}_{11}$. From Theorem 3, we see that Out_{11} is isomorphic to $5 \times 2.S_4$. There is only one conjugacy class of such sugroups in $GL_2\mathbf{F}_{11}$, because $GL_2\mathbf{F}_{11}$ is $5{\times}2.L_2(11).2$, and from the Atlas we see that $L_2(11).2$ contains only one conjugacy class of subgroups isomorphic to S_4. So, choosing new generators for 11_+^{1+2} if necessary, any subgroup of $GL_2\mathbf{F}_{11}$ which is isomorphic to $5 \times 2.S_4$ will do. Let $\mathcal{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} -1 & -3 \\ -3 & 1 \end{pmatrix}$, $\mathcal{D} = \begin{pmatrix} 3 & 5 \\ 4 & -4 \end{pmatrix}$, $\mathcal{E} = \begin{pmatrix} 4 & 5 \\ -3 & -4 \end{pmatrix}$ and $\mathcal{F} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. Then the subgroup of $GL_2\mathbf{F}_{11}$ generated by these six matrices is isomorphic to $5 \times 2.S_4$: for $\mathcal{F}$ has order 5 and is central, $\mathcal{A}$ is central, and the other relations are:

$$\begin{array}{ccc} \mathcal{A}^2 = \mathcal{D}^3 = \mathcal{E}^2 = I & & \mathcal{CB} = \mathcal{ABC} & \\ \mathcal{B}^2 = \mathcal{C}^2 = \mathcal{A} & \mathcal{DB} = \mathcal{BCD} & & \mathcal{DC} = \mathcal{BD} \\ & \mathcal{EB} = \mathcal{ACE} & \mathcal{EC} = \mathcal{ABE} & \mathcal{ED} = \mathcal{D}^{-1}\mathcal{E} \end{array}.$$

This corresponds to taking generators (1 2)(3 4), (1 3)(2 4), (2 3 4) and (2 3) for S_4.

Let $\mathcal{H}$ be $\mathcal{EF}$. We will use the normal series

$$1 \lhd \langle\mathcal{B}\rangle \lhd \langle\mathcal{B},\mathcal{C}\rangle \lhd \langle\mathcal{B},\mathcal{D}\rangle \lhd \langle\mathcal{B},\mathcal{D},\mathcal{H}\rangle = Out_{11}$$

in order to calculate the invariant classes. In the following results we obtain the ring of classes fixed by each successive term in the normal series. Since the series is normal, each group in the series acts on the fixed ring of the previous group. Theorem 9 is a trivial corollary of Proposition 13 below. We begin by concentrating on the ring generated by α_1 and α_2.

Proposition 10 *Let S be $\mathbf{F}_{11}[\alpha_1,\alpha_2]/(\alpha_1\alpha_2^{11} - \alpha_1^{11}\alpha_2)$. Then the subring of elements which are fixed by $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ is generated by*

$$\begin{aligned} E &= \alpha_1^6 - 2\alpha_1^5\alpha_2 - 5\alpha_1^4\alpha_2^2 - 5\alpha_1^2\alpha_2^4 + 2\alpha_1\alpha_2^5 + \alpha_2^6 \\ G &= \alpha_1^8 + \alpha_1^7\alpha_2 - 4\alpha_1^6\alpha_2^2 + 4\alpha_1^5\alpha_2^3 - 4\alpha_1^3\alpha_2^5 - 4\alpha_1^2\alpha_2^6 - \alpha_1\alpha_2^7 + \alpha_2^8, \end{aligned}$$

and is $\mathbf{F}_{11}[E,G]/(E^4 - G^3)$.

In the following lemma we start to prove this proposition by finding the subring fixed by $\mathcal{B}$. The proof of this lemma is typical of the invariant calculations to come, and we give the proof in some detail.

Lemma 11 *In Proposition 10, the subring S^1 of elements which are fixed by $\mathcal{B}$ is generated by*

$$A = \alpha_1^2 + \alpha_2^2 \qquad B = \alpha_1^4 + 5\alpha_1^2\alpha_2^2 + \alpha_2^4 \qquad C = 4(\alpha_1\alpha_2^3 - \alpha_1^3\alpha_2),$$

and is $\mathbf{F}_{11}[A,B,C]/(3B^2C - C^3,\ A^4 - B^2 - C^2)$.

Proof: Let V be a two-dimensional $\mathbf{F}_{11}$-vector space, and let α_1 and α_2 be a basis for its dual V^*. Let Σ be the symmetric algebra $\mathbf{F}_{11}[\alpha_1, \alpha_2]$ of V^*. Then S is the quotient $\Sigma/(\alpha_1\alpha_2^{11} - \alpha_1^{11}\alpha_2)$. The natural action of $GL(V)$ on V^* induces an action on Σ. This induces in turn an action on S, because matrices act on $\alpha_1\alpha_2^{11} - \alpha_1^{11}\alpha_2$ as multiplication by their determinant. This is the action described in Theorem 6.

We want the elements in S that are fixed by $\mathcal{B} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, which sends α_1 to α_2 and α_2 to $-\alpha_1$. We start by choosing new generators for S which are eigenvectors for $\mathcal{B}$. The characteristic polynomial of $\mathcal{B}$ is $X^2 + 1$, and since $\mathbf{F}_{11}$ does not contain a square root of -1, we extend the field to $\mathbf{F}_{121}$. Let i be a square root of -1 in $\mathbf{F}_{121}$, and denote by $\hat{V}^*$, $\hat{\Sigma}$ and $\hat{S}$ the obvious extensions. Setting $\beta_1 = \alpha_1 - i\alpha_2$ and $\beta_2 = \alpha_1 + i\alpha_2$, we have a new basis for $\hat{V}^*$ with $\mathcal{B}\beta_1 = i\beta_1$ and $\mathcal{B}\beta_2 = -i\beta_2$. Then $\hat{\Sigma} = \mathbf{F}_{121}[\beta_1, \beta_2]$, and every monomial $\beta_1^n\beta_2^m$ is an eigenvector with eigenvalue i^{n-m}.

The fixed elements in $\hat{\Sigma}$ are spanned by the $\beta_1^n\beta_2^m$ for which $n \equiv m \bmod 4$, and are therefore generated by $\gamma_1 = \beta_1^4$, $\gamma_2 = \beta_1\beta_2$ and $\gamma_3 = \beta_2^4$. The relation $\gamma_2^4 = \gamma_1\gamma_3$ is clear from the definitions. Since the β_i are algebraically independent in $\hat{\Sigma}$, this is the only relation. Now turn to $\hat{S}$: certainly (the images of) the β_i generate $\hat{S}$, and the fixed subring $\hat{S}^1$ contains the γ_i. The γ_i generate $\hat{S}^1$ too: for $\beta_1^{12} - \beta_2^{12} = 2i(\alpha_1\alpha_2^{11} - \alpha_1^{11}\alpha_2)$, and so the quotient map from $\hat{\Sigma}$ to $\hat{S}$ is a bijection when restricted to the span of those $\beta_1^n\beta_2^m$ with $m < 12$.

So $\hat{S}^1$ is generated by γ_1, γ_2 and γ_3, which satisfy the relations $\gamma_1^3 = \gamma_3^3$ and $\gamma_2^4 = \gamma_1\gamma_3$. We show that these two relations are sufficient. Let J be the ideal in $\hat{\Sigma}$ consisting of the relations in $\hat{S}$: then J is generated by $\gamma_1^3 - \gamma_3^3$. This generator lies in $\hat{\Sigma}^1$, the subring of fixed elements in $\hat{\Sigma}$. To show that our relations are sufficient we have to show that $\gamma_1^3 - \gamma_3^3$ generates the ideal $\hat{\Sigma}^1 \cap J$ in $\hat{\Sigma}^1$. This means that for every x in $\hat{\Sigma}$ such that $(\gamma_1^3 - \gamma_3^3)x$ is fixed, we must prove that there is a fixed y such that $(\gamma_1^3 - \gamma_3^3)x = (\gamma_1^3 - \gamma_3^3)y$.

To prove this, let T be the subspace of $\hat{\Sigma}$ with basis those monomials $\beta_1^n\beta_2^m$ such that $n \not\equiv m \bmod 4$. Then T is additively a complement of $\hat{\Sigma}^1$ in $\hat{\Sigma}$, and if $x \in T$ is such that $(\beta_1^{12} - \beta_2^{12})x \in \hat{\Sigma}^1$, then $(\beta_1^{12} - \beta_2^{12})x = 0$.

The intersection of S and $\hat{S}^1$ is the fixed points of $\hat{S}^1$ under the action of the Galois group of $\mathbf{F}_{121}$ over $\mathbf{F}_{11}$. The non-trivial automorphism switches β_1 and β_2, fixes γ_2 and switches γ_1 and γ_3. So if we define $A = \gamma_2$, $B = \frac{1}{2}(\gamma_1 + \gamma_3)$ and $C = \frac{1}{2i}(\gamma_1 - \gamma_3)$ then A, B and C all lie in S^1. Since $\hat{S}^1$ is generated over $\mathbf{F}_{121}$ by A, B and C subject to the relations $C^3 = 3B^2C$ and $A^4 = B^2 + C^2$, it follows that S^1 is generated over $\mathbf{F}_{11}$ with the same generators and relations. ∎

We now continue with the proof of Proposition 10; having worked through one step in detail, we now only give the highlights.

Proof of Proposition 10: The matrix $\mathcal{C} = \begin{pmatrix} -1 & -3 \\ -3 & 1 \end{pmatrix}$ sends α_1 to $-\alpha_1 - 3\alpha_2$ and α_2 to $-3\alpha_1 + \alpha_2$, and so sends A to $-A$, sends B to $-5B - 3C$, and C to $-3B + 5C$. The eigenvectors are $\beta_1 = B - 2C$, $\beta_2 = B - 5C$ and $\beta_3 = A$, with eigenvalues 1, -1 and -1 respectively. Then S^1 is the quotient of $\mathbf{F}_{11}[\beta_1, \beta_2, \beta_3]$ by the relations $\beta_2^2 = -3\beta_1^2 + 4\beta_3^4$ and $\beta_1\beta_2^2 = \beta_1^3$. The subring S^2 of elements in S^1 fixed by $\mathcal{C}$ is therefore spanned by the $\beta_1^n\beta_2^m\beta_3^\ell$ for $m \equiv \ell$ mod 2, and generated by $\gamma_1 = \beta_1$, $\gamma_2 = \beta_2^2$, $\gamma_3 = \beta_2\beta_3$ and $\gamma_4 = \beta_3^2$. The relation $\gamma_3^2 = \gamma_2\gamma_4$ follows from the definition, and the relations in S^1 become $\gamma_2 = -3\gamma_1^2 + 4\gamma_4^2$ and $\gamma_1\gamma_2 = \gamma_1^3$. These relations are sufficient: consider the complementary subspace spanned by the $\beta_1^n\beta_2^m\beta_3^\ell$ with $m \not\equiv \ell$ mod 2.

Generator γ_2 is redundant, so we eliminate it and put $D = \gamma_1$, put $E = \gamma_3$ and $F = \gamma_4$ to get S^2 isomorphic to the quotient of $\mathbf{F}_{11}[D, E, F]$ by the relations $E^2 = 4F^3 - 3D^2F$ and $D^3 = DF^2$. In terms of α_1 and α_2, we have $F = \alpha_1^4 + 2\alpha_1^2\alpha_2^2 + \alpha_2^4$ and

$$\begin{aligned} D &= \alpha_1^4 - 3\alpha_1^3\alpha_2 + 5\alpha_1^2\alpha_2^2 + 3\alpha_1\alpha_2^3 + \alpha_2^4 \\ E &= \alpha_1^6 - 2\alpha_1^5\alpha_2 - 5\alpha_1^4\alpha_2^2 - 5\alpha_1^2\alpha_2^4 + 2\alpha_1\alpha_2^5 + \alpha_2^6. \end{aligned}$$

We now look for the elements also fixed by $\mathcal{D} = \begin{pmatrix} 3 & 5 \\ 4 & -4 \end{pmatrix}$: this sends α_1 to $3\alpha_1 + 5\alpha_2$ and α_2 to $4\alpha_1 - 4\alpha_2$. So $\mathcal{D}$ sends D to $5D - 5F$, fixes E and sends F to $4D + 5F$. The characteristic polynomial $X^2 + X + 1$ splits over $\mathbf{F}_{121}$ with roots $5 \pm 3i$. Eigenvectors are $\beta_1 = E$, $\beta_2 = D - 2iF$ and $\beta_3 = D + 2iF$, with eigenvalues 1, $5 + 3i$ and $5 - 3i$ respectively. Then $\hat{S}^2$ is the quotient of $\mathbf{F}_{121}[\beta_1, \beta_2, \beta_3]$ by the relations $\beta_3^3 = -\beta_2^3$ and $\beta_2^3 - \beta_3^3 = 4i\beta_1^2$. The fixed elements are spanned by the $\beta_1^n\beta_2^m\beta_3^\ell$ with $m \equiv \ell$ mod 3 and so generated by $\gamma_1 = \beta_1$, $\gamma_2 = \beta_2^3$, $\gamma_3 = \beta_2\beta_3$ and $\gamma_4 = \beta_3^3$. The relation $\gamma_3^3 = \gamma_2\gamma_4$ is immediate from these definitions, and the relations in $\hat{S}^2$ become $\gamma_4 = -\gamma_2$ and $\gamma_2 - \gamma_4 = 4i\gamma_1^2$. Sufficiency: consider the subspace spanned by the $\beta_1^n\beta_2^m\beta_3^\ell$ with $m \not\equiv \ell$.

We eliminate the redundant generators γ_2 and γ_4. Set $G = -2\gamma_3$ and recall that $\gamma_1 = E$. These are both defined over $\mathbf{F}_{11}$, and so the result is proved. ■

We now use the result of Proposition 10, together with the decomposition of Corollary 7 and the isomorphism Φ of Corollary 8 to find the classes in the cohomology of 11_+^{1+2} which are fixed by $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$.

Proposition 12 *The subring R^3 of $\mathrm{H}^*(11_+^{1+2}, \mathbf{Z})$ of classes which are fixed by the actions of $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ is generated by E, G, $e = \Phi E$, $g = \Phi G$, κ, ζ and the θ_j. All generators have additive order 11, apart from ζ, which has 11^2. The classes e, g and θ_j square to zero. The other relations are:*

$$\begin{array}{llll} Eg = eG & E^3 = -G\kappa & \multirow{2}{*}{$E^2G = \kappa^2$} & G^2 = -E\kappa \\ eg = 0 & E^2e = -g\kappa & & Gg = -e\kappa \end{array}$$

$$E\theta_j = G\theta_j = \kappa\theta_j = e\theta_j = g\theta_j = \theta_k\theta_j = 0.$$

Proof: The matrices $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ all have determinant 1, and so fix κ, ζ and the θ_j. Hence V_3 is contained in R^3. Since V_1 is graded by powers of ζ, Proposition 10 implies that the fixed classes in V_1 are the polynomials in E, G and ζ. Since $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ all commute with Φ, the fixed classes in V_2 are precisely the images under Φ of the fixed classes in V_1. Hence R^3 is generated by E, G, e, g, κ, ζ and the θ_j.

Relations: From Proposition 10 we get $E^4 = G^3$, and it follows from Corollary 8 that $Eg = Ge$ and $E^3e = G^2g$. Since $\alpha_i\nu_1\nu_2 = 0$, we have $eg = 0$. We know that κ^2, $E\kappa$ and $G\kappa$ must be polynomials in E and G: comparing degrees, we see that they are scalar multiples of E^2G, G^2 and E^3 respectively. Comparing coefficients of highest powers of α_1 we have $\kappa^2 = E^2G$, $E\kappa = -G^2$ and $G\kappa = -E^3$. Applying Φ we obtain $e\kappa = -Gg$ and $g\kappa = -E^2e$. The relations $E^4 = G^3$ and $E^3e = G^2g$ are redundant, because they follow from the two ways of evaluating $EG\kappa$, $Ge\kappa$ respectively.

Sufficiency: Any polynomial in the generators can be decomposed into a sum of classes in V_1, V_2 and V_3, and so we must show that we have all relations inside these three submodules. All elements in V_3 are fixed, so we are safe there. Since degree of ζ grades V_1, all the relations there are accounted for by $E^4 = G^3$ and $11^2\zeta = 11E = 11G = 0$. Since Φ is an isomorphism from V_1 to V_2, extra relations inside V_2 would imply extra relations inside V_1. ∎

Proposition 13 *The ring* $\mathrm{H}^{\mathrm{inv}}$ *of classes in* $\mathrm{H}^*(11_+^{1+2}, \mathbf{Z})$ *which are fixed by* $\mathcal{B}$, $\mathcal{D}$ *and* $\mathcal{H}$ *is generated by* κ, $\eta = \zeta^{10}$, $\phi_i = \theta_i\zeta^{10-i}$, *and*

$$
\begin{aligned}
\alpha &= \left(\alpha_1^6 - 2\alpha_1^5\alpha_2 - 5\alpha_1^4\alpha_2^2 - 5\alpha_1^2\alpha_2^4 + 2\alpha_1\alpha_2^5 + \alpha_2^6\right)\zeta^2 \\
\mu &= \left((\alpha_1^5 - 2\alpha_1^4\alpha_2 - 5\alpha_1^3\alpha_2^2 - 5\alpha_1\alpha_2^4 + 2\alpha_2^5)\nu_1 + \alpha_2^5\nu_2\right)\zeta \\
\beta &= \left(\alpha_1^8 + \alpha_1^7\alpha_2 - 4\alpha_1^6\alpha_2^2 + 4\alpha_1^5\alpha_2^3 - 4\alpha_1^3\alpha_2^5 - 4\alpha_1^2\alpha_2^6 - \alpha_1\alpha_2^7 + \alpha_2^8\right)\zeta^6 \\
\nu &= \left((\alpha_1^7 + \alpha_1^6\alpha_2 - 4\alpha_1^5\alpha_2^2 + 4\alpha_1^4\alpha_2^3 - 4\alpha_1^2\alpha_2^5 - 4\alpha_1\alpha_2^6 - \alpha_2^7)\nu_1 + \alpha_2^7\nu_2\right)\zeta^5.
\end{aligned}
$$

All generators have additive order 11*, apart from* η*, which has* 11^2*; the generators* μ, ν *and* ϕ_j *square to zero; the other relations are:*

$$
\begin{array}{llll}
\alpha\nu = \beta\mu & \alpha^3 = -\beta\kappa & \multirow{2}{*}{$\alpha^2\beta = \kappa^2\eta$} & \beta^2 = -\alpha\kappa\eta \\
\mu\nu = 0 & \alpha^2\mu = -\nu\kappa & & \beta\nu = -\mu\kappa\eta
\end{array}
$$
$$
\alpha\phi_i = \beta\phi_i = \kappa\phi_i = \mu\phi_i = \nu\phi_i = \phi_j\phi_i = 0.
$$

Proof: We need the classes in R^3 which are fixed by $\mathcal{H}$. Recall that $\mathcal{H}$ is $\mathcal{EF} = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$. This has determinant 2, which has multiplicative order 10 in $\mathbf{Z}/11$ and 110 in $\mathbf{Z}/121$. So κ is fixed, θ_j is sent to $2^j\theta_j$, and ζ to $2^{11}\zeta$. For degree reasons, E and G must be eigenvectors; comparing coefficients of highest powers of α_1, we get $E \mapsto 2^{-2}E$ and $G \mapsto 2^4E$. Hence $e \mapsto 2^{-1}e$ and $g \mapsto 2^5g$. Therefore all generators of R^3 are eigenvectors.

The fixed classes in V_3 are spanned by 1, $\zeta^{10\ell}$, $\kappa\zeta^{10\ell}$ and $\theta_j\zeta^{10\ell-j}$. Hence they are generated by κ, $\eta = \zeta^{10}$ and $\phi_j = \theta_j\zeta^{10-j}$. From Proposition 12, the intersection $V_1 \cap R^3$ is the module on E, G, EG, E^2 and κ^2 over the ring generated by κ and ζ. Hence the invariant classes in V_1 are the module on $E\zeta^2$, $G\zeta^6$, $EG\zeta^8$, $E^2\zeta^4$ and κ^2 over the ring generated by κ and ζ^{10}. Therefore we need two extra generators $\alpha = E\zeta^2$ and $\beta = G\zeta^6$ to account for V_1. Turning to V_3, we have $\mathcal{H}\Phi(x) = 2\Phi(\mathcal{H}x)$, and so $\mathcal{H}\Phi(x) = \Phi(\mathcal{H}(x\zeta))$. Hence Φx is fixed if and only if $x\zeta$ is, and so the invariant classes in V_2 are the module on $e\zeta$, $g\zeta^5$, $Ge\zeta^7$, $Ee\zeta^3$ and $\Phi(\kappa^2)\zeta^9 = EGe\zeta^9$ over the ring generated by κ and ζ^{10}. We set $\mu = e\zeta$ and $\nu = g\zeta^5$, and this completes our set of generators: note that $\Phi(\kappa^2)\zeta^9 = \alpha\beta\mu$. The relations are clearly true and sufficient, by comparison with the relations in Proposition 12. ■

This completes the proof of Theorem 9. We round off this section with a result that expresses the 11-primary cohomology of J_4 as a simply-defined subring of a ring with three generators.

Theorem 14 *Let R be a commutative graded $\mathbf{Z}_{(11)}$-algebra, generated by A and Z in degree 2 and M in degree 3, subject to the relations $M^2 = 0$ and $11^2 Z = 11A = 11M = 0$. Then there is a monomorphism which embeds $\mathrm{H}^*(J_4, \mathbf{Z})_{(11)}$ in R as follows:*

$$\begin{array}{llll} \alpha \mapsto A^6 Z^{22} & \beta \mapsto A^8 Z^{66} & \multirow{2}{*}{$\phi_j \mapsto 11 Z^{110-10j}$} & \kappa \mapsto -A^{10} \\ \mu \mapsto A^5 M Z^{11} & \nu \mapsto A^7 M Z^{55} & & \eta \mapsto Z^{110} \ . \end{array}$$

Proof: This map is easily seen to be well-defined, so we only have to show that it is injective. The ring $\mathrm{H}^*(J_4, \mathbf{Z})_{(11)}$ has the additive structure of the direct sum of three modules: the free $\mathbf{Z}_{(11)}$-module generated by 1; the free $\mathbf{Z}/11^2$-module on the η^ℓ for $\ell > 0$; and the $\mathbf{F}_{11}$-vector space with basis as follows:

$$\begin{array}{cccccc} \kappa^k\eta^\ell \text{ for } k > 0 & \alpha\kappa^k\eta^\ell & \beta\kappa^k\eta^\ell & \alpha\beta\kappa^k\eta^\ell & \alpha^2\kappa^k\eta^\ell & \\ \alpha\beta\mu\kappa^k\eta^\ell & \mu\kappa^k\eta^\ell & \nu\kappa^k\eta^\ell & \beta\mu\kappa^k\eta^\ell & \alpha\mu\kappa^k\eta^\ell & \phi_j\eta^\ell. \end{array}$$

It is straightforward to check that these generators all map to different monomials of the correct additive order, and the result is proved. ■

Notice that R is isomorphic to the $\mathbf{Z}_{(11)}$-cohomology ring of $C_{11} \times C_{11^2}$. It should also be noted that there does not exist such a nice embedding for $\mathrm{H}^*(p_+^{1+2}, \mathbf{Z})_{(p)}$.

3 The 3-local cohomology

We now apply the same methods to the 3-primary cohomology of J_4. Calculating the invariant classes is much more straightforward than in the $p = 11$ case, but this time not all invariant classes are stable. We prove:

Theorem 15 *The ring* $\mathrm{H}^*(J_4, \mathbf{Z})_{(3)}$ *is generated by* α *in degree* 16*,* ξ *in degree* 12 *and* ϖ *in degree* 27*. The only relations are* $3^2\xi = 3\alpha = 3\varpi = 0$ *and* $\varpi^2 = 0$*.*

By Theorem 1, the 3-local cohomology is the subring of stable classes in the cohomology of $P = 3_+^{1+2}$. Since all stable classes are invariant, we start by calculating $\mathrm{H}^{\mathrm{inv}}$, the ring of classes fixed by Out_3.

From Theorem 3, the order of Out_3 is 2^4, and it is therefore a Sylow 2-subgroup of $GL_2\mathbf{F}_3$. As Sylow subgroups are unique up to conjugacy, any one will do. Write $\mathcal{J} = \left(\begin{smallmatrix} 1 & -1 \\ -1 & -1 \end{smallmatrix}\right)$, $\mathcal{K} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\mathcal{L} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. Then $\mathcal{J} = \mathcal{L}^2$, and the subgroup generated by $\mathcal{K}$ and $\mathcal{L}$ has order 2^4: for $\mathcal{K}$ has order 4, $\mathcal{L}$ has order 8, $\mathcal{L}^4 = \mathcal{K}^2$, and $\mathcal{L}\mathcal{K} = \mathcal{K}\mathcal{L}^3$. Hence Out_3 is generated by $\mathcal{K}$ and $\mathcal{L}$ (up to choice of generators for P), and a normal series for Out_3 is

$$1 \triangleleft \langle \mathcal{J} \rangle \triangleleft \langle \mathcal{J}, \mathcal{K} \rangle \triangleleft \langle \mathcal{K}, \mathcal{L} \rangle = Out_3.$$

We now obtain the ring of classes fixed by each successive term in this normal series in a sequence of lemmas leading up to Proposition 17, where $\mathrm{H}^*(3_+^{1+2}, \mathbf{Z})^{\mathrm{inv}}$ is obtained. After this we address the problem of finding the stable classes.

Recall from Theorem 6 that the integral cohomology of 3_+^{1+2} is generated by α_1, α_2, ν_1, ν_2, κ and ζ.

Lemma 16 *The ring* R^2 *of classes which are fixed by* $\mathcal{J}$ *and* $\mathcal{K}$ *is generated by* κ*,* ζ *and* $c = \alpha_1^3\nu_1 - \alpha_1\alpha_2^2\nu_1 + \alpha_2^3\nu_2$*. The only relations are* $3^2\zeta = 3\kappa = 3c = 0$ *and* $c^2 = 0$*.*

Proof: Since $\mathcal{J}$ and $\mathcal{K}$ have determinant 1, they fix κ and ζ, and commute with the map Φ of Corollary 8. Let T be the ring generated by κ and ζ, and S the ring generated by α_1 and α_2. Every class in T is fixed, and we show below that the fixed classes in S are generated by κ^2 and κ^3. Hence, in the decomposition of Corollary 7, the intersection $R^2 \cap V_1$ is the T-module on κ^2. Every class in V_3 is fixed and by Corollary 8, $R^2 \cap V_2$ is the T-module on $c = \Phi\kappa^2$: so we have a generating set. The relations follow immediately, and allow us to decompose any polynomial in the generators into a sum of classes in V_1, V_2 and V_3. We certainly have all relations in V_1 and V_3, therefore also in V_2 by applying the isomorphism Φ: so the result is established.

It remains to prove that, in the ring S generated by α_1 and α_2, the fixed classes are generated by κ^2 and κ^3. We use the method of Lemma 11. Effectively, S is $\mathbf{F}_3[\alpha_1, \alpha_2]/(\alpha_1\alpha_2^3 - \alpha_1^3\alpha_2)$. Eigenvectors for $\mathcal{J}$ over $\mathbf{F}_9$ are $\beta_1 = \alpha_1 + \alpha_2 - i\alpha_2$ and $\beta_2 = \alpha_1 + \alpha_2 + i\alpha_2$ with eigenvalues i and $-i$ respectively. The ring S^1 of classes in S fixed by $\mathcal{J}$ is generated by $A = \alpha_1^2 - \alpha_1\alpha_2 - \alpha_2^2$ and $B = \alpha_1^4 - \alpha_1^3\alpha_2 - \alpha_2^4$ with the relation $B^2 = A^4$. Then $A\kappa = -B$ and $\kappa^2 = A^2$. Hence S^1 has a basis consisting of 1, $A\kappa^k$ and κ^ℓ, with $\ell > 1$. As $\mathcal{K}$ fixes κ and sends A to $-A$, the classes in S fixed by $\mathcal{J}$ and $\mathcal{K}$ are spanned by 1 and κ^ℓ for $\ell > 1$, as claimed. ∎

Proposition 17 *The ring* $\mathrm{H}^*(3^{1+2}_+, \mathbf{Z})^{\mathrm{inv}}$ *is generated by* κ, $\eta = \zeta^2$ *and* $d = (\alpha_1^3\nu_1 - \alpha_1\alpha_2^2\nu_1 + \alpha_2^3\nu_2)\zeta$. *The relations are* $3^2\eta = 3\kappa = 3d = 0$ *and* $d^2 = 0$.

Proof: κ is fixed as usual. $\mathcal{L}$ has determinant -1, and so $\zeta \mapsto -\zeta$. We also have $c \mapsto -c$, because $c = \Phi\kappa^2$ and $\mathcal{L}\Phi = -\Phi\mathcal{L}$. The fixed classes are therefore generated by κ, ζ^2 and $c\zeta$, and the relations are obvious. ∎

We must now determine which of the invariant classes are stable. As P has order 3^3, distinct conjugates of P can intersect with order 3^2, 3 or 1.

Proposition 18 *Let* G *be a finite group with Sylow* p*-subgroup* $P \cong p^{1+2}_+$, *and suppose that* $x \in \mathrm{H}^*(P, \mathbf{Z})$ *satisfies the stability condition for all* g *such that the intersection* $P \cap P^g$ *has order at least* p^2*: then* x *is stable.*

Proof: Trivial if the intersection has order 1. If order p, recall that x satisfies the stability condition for g when a certain pair of cohomology classes of $P \cap P^g$ are equal. This certainly implies equality after corestriction to P. Conversely, if x is "stable after corestriction" for all $g \in G$, then x is stable: inspect the proof in [**2**] of our Theorem 1. So x is stable if it is "stable after corestriction" for the g such that $P \cap P^g$ is cyclic of order p. But corestriction from C_p to p^{1+2}_+ factors through corestriction from C_p to $C_p \times C_p$, which is the zero map (away from degree zero, where all stability conditions are trivial). It is the zero map because restriction from $C_p \times C_p$ to C_p is surjective (split by inflation), restriction followed by corestriction is multiplication by the index p, and cohomology classes of $C_p \times C_p$ in positive degree have additive order p. ∎

Before studying intersections of order 3^2, we prove an important lemma which means that choosing new generators for P has no effect on the invariant classes.

Lemma 19 *Let* ψ *be an automorphism of* 3^{1+2}_+. *The induced cohomology automorphism* ψ^* *fixes every class in* $\mathrm{H}^*(3^{1+2}_+, \mathbf{Z})^{\mathrm{inv}}$.

Proof: From Proposition 17, the ring $\mathrm{H}^{\mathrm{inv}}$ is generated by κ, ζ^2 and $\Phi(\kappa^2\zeta)$. Define j by $\psi C = C^j$. The automorphism ψ^* fixes κ, multiplies ζ^2 by j^6 and multiplies $\Phi(\kappa^2\zeta)$ by j^2. Since $j \equiv \pm 1 \bmod 3$, we have $j^2 \equiv 1 \bmod 3$ and $j^6 \equiv 1 \bmod 9$. Hence ζ^2 and $\Phi(\kappa^2\zeta)$ are also fixed. ∎

All subgroups of $P \cong 3^{1+2}_+$ of order 3^2 have the structure $C_3 \times C_3$, and contain the centre. They are all normal, but are permuted transitively by the automorphism group. Recall that P has generators A, B and C, with C central.

Lemma 20 *Let $g \in J_4$ be such that P and P^g intersect with order 3^2, and let x be a class in $\mathrm{H}^*(P,\mathbf{Z})^{\mathrm{inv}}$. If P and P^g have the same centre then x is stable with respect to g. If they have different centres then x is stable with respect to g if and only if $\mathrm{Res}^P_{\langle B,C\rangle} x$ is fixed by the automorphism f of $\langle B,C\rangle$ which sends $B \leftrightarrow C$.*

Proof: Same centre: Since Lemma 19 allows us to choose new generators for 3^{1+2}_+ without affecting the invariant classes, we may assume that $P \cap P^g$ is the subgroup $\langle B,C\rangle$. Then gBg^{-1} is in P and gCg^{-1} is a power of C. Let ψ be an automorphism of P that sends B to gBg^{-1} and C to gCg^{-1}. Therefore including $\langle B,C\rangle$ in P^g and then conjugating by g is the same map to P as including in P and then applying ψ. In cohomology, this is the stability condition twisted by ψ^*. Since ψ^* fixes invariant classes, the result follows. Different centres: Again, we may assume that $P \cap P^g$ is $\langle B,C\rangle$, with gBg^{-1} now a power of C and gCg^{-1} an element of P. For ψ, pick an automorphism of P which sends B to gCg^{-1} and C to gBg^{-1}. Inclusion in P^g followed by conjugation is now the same map as f followed by inclusion in P followed by ψ, and the result follows as before. ∎

The importance of Lemma 19 here is that it allows us to choose automorphisms of 3^{1+2}_+ without having to consider whether they are induced by inner automorphisms of J_4. We now finish the proof of Theorem 15.

Proposition 21 *The stable classes in $\mathrm{H}^*(3^{1+2}_+,\mathbf{Z})$ are generated by $\alpha = \kappa\zeta^2$, $\xi = \zeta^2 - \kappa^3$ and $\varpi = (\alpha_1^3\nu_1 - \alpha_1\alpha_2^2\nu_1 + \alpha_2^3\nu_2)\zeta^3$. The relations are $3^2\xi = 3\alpha = 3\varpi = 0$ and $\varpi^2 = 0$.*

Proof: We have seen that the only $g \in J_4$ that impose non-trivial conditions for invariant classes to be stable are the g for which P and P^g have different centres and intersect with order 3^2. Such g do exist, because J_4 contains the Mathieu group M_{12}, which in turn contains $3^2{:}2S_4$. So an invariant class is stable when its restriction to $\langle B,C\rangle$ is fixed by the map f^*.

From Theorem 6, restriction to $\langle B,C\rangle$ sends κ to $-\beta^2$, sends η to $(\gamma^3 - \beta^2\gamma)^2$ and d to $\beta^3(\gamma^3 - \beta^2\gamma)\chi$. From Proposition 5 we see that f^* sends $\beta \leftrightarrow \gamma$ and χ to $-\chi$. Hence the classes $\kappa\eta$, $\eta - \kappa^3$, and $d\eta$ are stable: we claim that they generate all the stable classes.

Consider first an even-dimensional invariant class π whose restriction to $\langle B,C\rangle$ is fixed by f^*; then π is a polynomial in κ and η. We prove that π is a polynomial in $\kappa\eta$ and $\eta - \kappa^3$ by induction on degree. Subtracting powers of $\eta - \kappa^3$ if necessary, we assume $\pi = \kappa\eta\pi' + \ell\kappa^j$ for some $\ell \in \mathbf{F}_3$. Then the restriction of $\kappa\eta\pi'$ is divisible by $\beta^2\gamma^2$, whereas $\mathrm{Res}^P_{\langle B,C\rangle} \ell\kappa^j = (-1)^j\ell\beta^{2j}$. Hence ℓ is zero, for otherwise $\mathrm{Res}^P_{\langle B,C\rangle} \pi$ would contain a monomial β^{2j} without the corresponding γ^{2j} required to be fixed by f^*. Therefore $\pi = \kappa\eta\pi'$, and so $\mathrm{Res}^P_{\langle B,C\rangle} \pi'$ is fixed by f^*. Since π' has lower degree than π, it is a polynomial in $\kappa\eta$ and $\eta - \kappa^3$ by the inductive hypothesis, and so the inductive step is proved. The result is trivial in degree zero, and so $\kappa\eta$ and $\eta - \kappa^3$ do generate the even-dimensional stable classes. The odd-dimensional case is similar: no invariant class which includes a monomial of the form $d\kappa^j$ can be stable. We have a generating set for the stable classes, and the relations follow from Proposition 17. ∎

4 The Chern subring

Chern classes are a link between the ordinary representation theory and the integral cohomology of G. They are defined in the appendix of [**1**]. A representation ρ of a finite group G has Chern classes $c_i(\rho) \in \mathrm{H}^{2i}(G, \mathbf{Z})$ for $i \geqslant 0$, with $c_0\rho = 1$, and $c_i\rho$ is zero if i exceeds the degree of ρ. Two important properties are naturality, $c_i(f^!\rho) = f^*(c_i\rho)$, and the Whitney sum formula $c(\rho_1 \oplus \rho_2) = c(\rho_1)c(\rho_2)$. Here $c(\rho) = \sum_{i \geqslant 0} c_i\rho$ is the *total* Chern class of ρ.

The Chern subring of G, denoted $Ch(G)$, is the subring of $\mathrm{H}^*(G, \mathbf{Z})$ generated by all Chern classes in $\mathrm{H}^*(G, \mathbf{Z})$. In this section we prove:

Theorem 22 *The* 11*-primary part* $\mathrm{Ch}(J_4)_{(11)}$ *of the Chern subring is generated by* κ*,* η *and the* ϕ_i*. It is also generated by the Chern classes of a representation affording the irreducible character* χ_4 *of degree* $299,367$ *(Atlas notation).*

The 3*-primary part* $\mathrm{Ch}(J_4)_{(3)}$ *of the Chern subring is generated by* 3ξ*,* $3\xi^2$*,* ξ^3 *and* α^3*. It is also generated by the Chern classes of a representation affording the irreducible character* χ_2 *of degree* $1,333$*.*

Restriction commutes with taking Chern classes. So we restrict each representation of J_4 to a Sylow p-subgroup, decompose this restriction, and apply the Whitney formula.

The irreducible representations of p_+^{1+2} are ρ^{xy} for $0 \leqslant x, y \leqslant p-1$, and ρ^z for $1 \leqslant z \leqslant p-1$. They have characters

$$\begin{aligned} \chi^{xy}\colon A^rB^sC^t &\longmapsto \exp\{2\pi i(rx+sy)/p\} \\ \chi^{z}\colon A^rB^sC^t &\longmapsto \begin{cases} p\exp\{2\pi izt/p\} & r=s=0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The ρ^{xy} are 1-dimensional, and the ρ^z are induced from $\langle B, C\rangle$.

Theorem 23 *([**6**]) The total Chern classes of the irreducible representations of* p_+^{1+2} *are* $c(\rho^{xy}) = 1 + x\alpha_1 + y\alpha_2$ *and* $c(\rho^z) = 1 + \sum_{j=2}^{p-2} \frac{1}{p}\binom{p}{j} z^j\theta_j + \kappa + z^p\zeta$*.* ∎

We call the intersection of P with a conjugacy class of G a *G-conjugacy class* in P, and define a *G-class function* on P to be a function on P which is constant on the G-conjugacy classes. Then characters of G restricted to P are G-class functions.

Lemma 24 *Let P a Sylow p-subgroup of a finite group G. Then $\mathrm{Ch}(G)_{(p)}$ is the subring of $\mathrm{H}^*(P, \mathbf{Z})$ generated by the Chern classes of those representations of P whose characters are G-class functions.*

Proof: It is enough to show that every character ϕ of P which is a G-class function extends to a generalised character of G. From Brauer's Induction Theorem, a class function on G is a generalised character if its restriction to every elementary subgroup of G is a generalised character.

Extend ϕ to a class function ψ on G by defining ψ in the obvious way on p-elements (including the identity). If g has order mp^r where $p \nmid m$, then pick an s such that $m \mid s$ and $p^r \equiv 1 \bmod s$, and define $\psi(g) = \psi(g^s)$. Clearly ψ is a character after restriction to any subgroup which is the direct product of a p-group and a p'-group; but all elementary groups are of this form. ∎

We obtain a spanning set for those characters of P which are G-class functions. In p_+^{1+2}, let pC be the set of non-identity central elements, let pD be the non-central elements and pE all non-identity elements. From Theorem 3, the J_4-conjugacy classes in 11_+^{1+2} are 1, $11C$ and $11D$. In 3_+^{1+2} they are 1 and $3E$.

Lemma 25 *Let χ be a generalised character of p_+^{1+2} which is constant on pC and on pD. Then χ is in the $\mathbf{Z}$-span of χ^{00}, $\chi^D = \sum_{x,y=0}^{p-1} \chi^{xy}$ and $\chi^C = \sum_{z=1}^{p-1} \chi^z$. If χ is constant on pE then χ is in the $\mathbf{Z}$-span of χ^{00} and $\chi^E = \chi^D + p\chi^C$.*

So the J_4-class functions in 11_+^{1+2} are spanned by χ^{00}, χ^D and χ^C, and in 3_+^{1+2} by χ^{00} and χ^E. We have character tables

	1	$11C$	$11D$
χ^{00}	1	1	1
χ^D	121	121	0
χ^C	110	-11	0

	1	$3E$
χ^{00}	1	1
χ^E	27	0

.

Proof: First part: the automorphism group of p_+^{1+2} acts transitively on pC and on pD. Hence it fixes χ and χ^{00}, and acts transitively on the χ^{xy} (x and y not both zero) and on the χ^z. But χ is in the $\mathbf{Z}$-span of the χ^{xy} and the χ^z. The rest is obvious. ∎

Proposition 26 *Representations affording the characters χ^D and χ^C have total Chern classes $c(\rho^D) = 1 + \kappa^p$ and*

$$c(\rho^C) = (1+\kappa)^{p-1} - \zeta^{p-1} + \sum_{i=2}^{p-2} (-1)^{i+1} \frac{1}{p} \binom{p}{i} \theta_i \zeta^{p-i-1}.$$

Proof: We shall make repeated use of the identity $\prod_{r=0}^{p-1}(X-r) \equiv X^p - X$ mod p. The Chern classes of ρ^{xy} and ρ^z are in Theorem 23. By the Whitney formula,

$$\begin{aligned} c(\rho^D) &= \prod_{x,y=0}^{p-1}(1 + x\alpha_1 + y\alpha_2) \\ &= \prod_{x=0}^{p-1}\left((1+x\alpha_1)^p - (1+x\alpha_1)\alpha_2^{p-1}\right) \\ &= (1-\alpha_2^{p-1})^p - (1-\alpha_2^{p-1})(\alpha_1^p - \alpha_1\alpha_2^{p-1})^{p-1}. \end{aligned}$$

Now, $\alpha_2(\alpha_1^p - \alpha_1\alpha_2^{p-1}) = \alpha_1^p\alpha_2 - \alpha_1\alpha_2^p$, which is zero. Hence

$$(1-\alpha_2^{p-1})(\alpha_1^p - \alpha_1\alpha_2^{p-1})^{p-1} = \alpha_1^{p(p-1)} - \alpha_1^{(p-1)^2}\alpha_2^{p-1}.$$

Therefore

$$\begin{aligned} c(\rho^D) &= (1-\alpha_2^{p-1})^p - \alpha_1^{p(p-1)} + \alpha_1^{(p-1)^2}\alpha_2^{p-1} \\ &= 1 + \kappa^p, \end{aligned}$$

since, by induction on r,

$$\kappa^r = (-1)^r(\alpha_1^{r(p-1)} - \alpha_1^{(r-1)(p-1)}\alpha_2^{p-1} + \alpha_2^{r(p-1)}) \quad \text{for } r \geqslant 2.$$

For ρ^C, the Whitney formula gives

$$\begin{aligned} c(\rho^C) &= \prod_{z=1}^{p-1}\left(1 + \sum_{i=2}^{p-2}\frac{1}{p}\binom{p}{i}z^i\theta_i + \kappa + z^p\zeta\right) \\ &= (1+\kappa)^{p-1} - 1 + \prod_{z=1}^{p-1}(1+z^p\zeta) + \sum_{i=2}^{p-2}\frac{1}{p}\binom{p}{i}\sum_{j=0}^{p-2}a_{ij}\theta_i\zeta^j, \end{aligned}$$

where a_{ij} is the coefficient of XY^j in $\prod_{z=1}^{p-1}(1 + z^iX + zY)$. There are no terms $\kappa^i\zeta^j$, with i and j both positive, because

$$\prod_{z=1}^{p-1}(1 + \kappa + z^p\zeta) \equiv (1+\kappa)^{p-1} - \zeta^{p-1} \pmod{p}.$$

Since $X^p \equiv X$ mod p, we see that 1^p, 2^p, …, $(p-1)^p$ are $p-1$ distinct $(p-1)$-roots of 1 in $\mathbf{Z}/p^2$, and these two facts imply that

$$\prod_{z=1}^{p-1}(X - z^p) \equiv X^{p-1} - 1 \pmod{p^2}.$$

Therefore $\prod_{z=1}^{p-1}(1+z^p\zeta) = 1 - \zeta^{p-1}$. Lemma 27 below says that $a_{ij} \equiv 0 \pmod{p}$ unless $i+j = p-1$, in which case $a_{ij} \equiv (-1)^{i-1}$. The result is proved. ■

Lemma 27 *Let a_{ij} be the coefficient of XY^j in $\prod_{z=1}^{p-1}(1+z^iX+zY)$, where $1 \leqslant i \leqslant p-1$ and $0 \leqslant j \leqslant p-2$. Then*

$$a_{ij} \equiv \begin{cases} (-1)^{i-1} & \text{if } i+j=p-1 \\ 0 & \text{otherwise} \end{cases} \pmod p.$$

Proof: The result is certainly true for $i=1$, because a_{1j} is the coefficient of XY^j in $1-(X+Y)^{p-1}$. It is also true for $j=p-2$, since $a_{i,p-2}$ is the coefficient of XY^{p-2} in $\prod_{z=1}^{p-1}(z^iX+zY)$, which means that $a_{i,p-2}=(p-1)!\sum_{z=1}^{p-1} z^{i-1}$.

Otherwise, if $j \leqslant p-3$ and $i \geqslant 2$, then $a_{ij}+a_{i-1,j+1} = b_{j+1}\sum_{z=1}^{p-1} z^{i-1}$. Here b_j is the coefficient of Y^j in $\prod_{z=1}^{p-1}(1+zY)$, with a_{ij} corresponding to the terms $z_1^i z_2 \dots z_{j+1}$, and $a_{i-1,j+1}$ to the $z_1^{i-1}z_2\dots z_{j+2}$. But for these i, the sum $\sum_{z=1}^{p-1} z^{i-1}$ is zero mod p. Therefore $a_{ij}+a_{i-1,j+1} \equiv 0 \pmod p$. ∎

Proof of Theorem 22: We know the Chern classes of ρ^D and ρ^C from Proposition 26. For $p=3$, a representation affording χ^E has total Chern class $(1+\kappa^3)\left((1+\kappa)^2-\zeta^2\right)^3 = 1-3\xi+3\xi^2-\xi^3-\alpha^3$. Therefore by Lemma 24, we have the claimed generators for the Chern subring. It may easily be shown that the Chern classes of the stated representations of J_4 also generate. ∎

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