A GROUP-THEORETIC CONSEQUENCE OF THE DONALD–FLANIGAN CONJECTURE

Murray Gerstenhaber and David J. Green

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Abstract

For a finite group G and a prime p dividing the order of G, Donald and Flanigan conjecture that the group algebra $\overline{\mathbb{F}}_pG$ can be deformed into a semisimple (hence rigid) algebra. We demonstrate that this implies that for some element g of G, the centralizer $C_G(g)$ of g in G has a normal subgroup of index p. The method is to observe that the Donald–Flanigan deformation must be a jump. This implies that there is a non-trivial class in $\mathrm{H}^1(\mathbb{F}_pG, \mathbb{F}_pG)$; therefore this Hochschild cohomlogy group must be non-trivial. Using a standard result linking Hochschild and group cohomology one sees that some $\mathrm{H}^1(C_G(g), \mathbb{F}_p)$ must be non-zero. The result follows immediately.

1 Introduction

Let G be a finite group, p be a prime dividing its order $|G|$, and k be an algebraically closed field of characteristic p. The Donald–Flanigan conjecture [2] asserts that the group algebra kG, which is not semisimple, can nevertheless be deformed to a semisimple algebra. It is, in a sense, a p-modular version of Maschke's theorem. M. Schaps [16] refined the conjecture to assert that the deformation could moreover be so chosen that the matric blocks of the resulting algebra had the same dimensions as those of $\mathbb{C}G$, the complex group algebra. If true, the conjecture may have important consequences for modular representation theory. We show here that it has as a corollary a purely group theoretic property with a simple statement: there must exist in G an element g whose centralizer $C_G(g)$ has a normal subgroup of index p. (Equivalently, the index of the derived group, $C_G(g)'$, is divisible by p.) We have recently learned that, with knowledge of this implication, Fleischmann, Janiszczak and Lempken [6] have in fact proven a somewhat stronger assertion with the virtue that its proof can be reduced to the case where the group is simple: If G is a finite group of order divisible by a prime p then G contains a p-singular element g whose p-part is not contained in the commutator subgroup of $C_G(g)$. Their proof consists of an efficient examination of all simple groups and they cite the result in fact as new evidence for the Donald–Flanigan conjecture.

We will call a semisimple deformation of kG a "DF deformation". Such a deformation need not be unique; there may be others in addition to a "Schaps" deformation with the correct components.

It was originally an observation of Knörr that the element g whose existence our Corollary asserts may in fact be assumed to be p-singular, *i.e.*, to have order divisible by p. On the other hand it need not be a p-element (one whose order is a power of p). For the sporadic simple Janko group J_4 and $p = 3$, no 3-element serves. The Rudvalis, Thompson and Tits groups Ru, Th and ${}^2F_4(2)'$ also produce counterexamples, for the primes 3, 5 and 3 respectively. In all four cases the Sylow p-subgroups are extraspecial of order p^3 and exponent p. Curiously, in M_{23} with $p = 2$ there is a suitable g which is a product of three disjoint 7-cycles and hence has order prime to 2.

The present conjecture asserts precisely, as we shall see, that the Hochschild cohomology group $\mathrm{H}^n(\mathbb{F}_pG, \mathbb{F}_pG)$ of $\mathbb{F}_pG$ with coefficients in itself as a bimodule is non-trivial for $n = 1$. Of course, the Donald–Flanigan conjecture requires that it be non-trivial for $n = 2$, else there could be no deformation. But it is implicit that a DF deformation be a jump deformation, and this implies, as we shall see, that also $\mathrm{H}^1 \neq 0$. The concept of a jump deformation is fundamental to both parts of what we show. We recapitulate that theory, but first review briefly the present state of the DF and Schaps conjectures.

2 History

The remarkable Donald–Flanigan conjecture was proven by them only for abelian groups. The first subsequent progress was not until 1988 when Schaps [18] solved the problem for groups G with cyclic Sylow p-subgroups. The theorem was proven by the first author and Schaps [13] for G with an abelian normal Sylow p-subgroup, by Erdmann and Schaps [5] for characteristic $p = 2$ when the Sylow 2-subgroups of G are dihedral, and most recently by the first author and Schaps [14] for the symmetric groups. It is natural to extend the problem by considering not only deformations of the entire group algebra kG but of blocks of that algebra, *i.e.*, its indecomposable direct summands B. To each is associated a defect group $\delta(B)$, but a deformation of the group algebra of $\delta(B)$ does not automatically give a deformation of B nor conversely. Nevertheless, to date the few cases where B has been proven to have a semisimple deformation are precisely the cases where $k\delta(B)$ has been proven to have one. This is partly a consequence of the observation that for these B the endomorphism ring of an indecomposable projective of B is isomorphic to the group algebra of $\delta(B)$, so deformation of B gives a deformation of $k\delta(B)$, *cf.* [4], [3] and [5]. They have shown for arbitrary p

that every tame block with dihedral defect group has a semisimple deformation.

Another direction was initiated by Michler [15] who asked when a DF deformation of a block could be lifted to a deformation which is generically a maximal order descending modulo p to the block. He proved this for a block with cyclic defect group. Independently, Schaps [17] proved a liftability result for p–p' metacyclic groups. A recent approach to the full problem, due to Schaps, is to consider liftable deformations of Coxeter groups as deformations of Hecke algebras. At present it appears that all groups of order less than 32 are known to have DF deformations except the extraspecial 3-group of order 3^3 and exponent 3. This group, which occurs in three of the examples in §1, may be presented as a central extension of $C_3 \times C_3$ by C_3, where if c generates the central C_3 and a, b descend to generators of the quotient $C_3 \times C_3$ we have $aba^{-1}b^{-1} = c$.

3 Jump deformations

Henceforth "δ" will denote the Hochschild coboundary operator. For an exposition of the algebraic deformation theory, *cf.* [12]; the original papers are [7, 8, 9, 10, 11]. (The observation that there can be no jump deformations when the first cohomology vanishes is already contained in [12] but for completeness we recapitulate the main ideas here.)

Let A be an algebra over a ring k (here not necessarily a field) with multiplication $\alpha\colon A \times A \to A$. A (formal) deformation of A is a power series $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$ where all α_i are k-bilinear maps $A \times A \to A$ (extended to be $k[[t]]$-bilinear) and which is formally associative, *i.e.*, where $\alpha_t(\alpha_t(a,b),c) = \alpha_t(a,\alpha_t(b,c))$, all a, b, $c \in A$. This is equivalent to having, for each n, the identity

$$\sum_{i+j=n;\ i,j>0} \{\alpha_i(\alpha_j(a,b),c) - \alpha_i(a,\alpha_j(b,c))\} = \delta\alpha_n(a,b,c),$$

where δ is the Hochschild coboundary operator. In particular, $\delta\alpha_1 = 0$, so $\alpha_1 \in Z^2 = Z^2(A,A)$, the group of Hochschild 2-cocycles. If $\alpha_1 = \cdots = \alpha_{n-1} = 0$ and $\alpha_n \neq 0$ then $\delta\alpha_n = 0$ and the deformation "begins" with α_n. The deformation α_t is *equivalent* to a second, $\alpha'_t = \alpha + t\alpha'_1 + t^2\alpha'_2 + \cdots$ if there exists $\Phi_t = 1(= \mathrm{id}_A) + t\phi_1 + t^2\phi_2 + \cdots$ where all ϕ_i are k-linear maps $A \to A$ such that $\alpha'_t(a,b) = \Phi_t^{-1}\alpha_t(\Phi_t a, \Phi_t b) = \Phi_t^{-1}\alpha_t(\Phi_t \otimes \Phi_t)(a \otimes b)$. This implies, in particular, that $\alpha'_1 = \alpha_1 + \delta\phi_1$. Thus, up to equivalence of deformations, only the class $[\alpha_1] \in \mathrm{H}^2(A,A)$ is significant, and by successive equivalences we may assume that α_t begins with an α_n not a coboundary. Denote the deformed algebra (whose underlying module is $A[[t]]$) with multiplication α_t by A_t. Write $k[[t]] = k_t$. We call α_t a *jump deformation* if $\alpha_{(1+u)t}$, considered as a deformation of α_t over $k_t[[u]]$, is equivalent to α_t. Intuitively, a deformation "infinitely close" to α_t is equivalent to it. There must then exist $\Psi_u = 1 + u\psi_1 + u^2\psi_2 + \cdots$, where the ψ_i are themselves now k_t-linear maps from A_t to A_t, such that $\alpha_{(1+u)t} = \Psi_u^{-1}\alpha_t(\Psi_u \otimes \Psi_u)$. Collecting powers of u, the left hand side has the form $\alpha_t + u\alpha_{t,1} + u^2\alpha_{t,2} + \cdots$, where $\alpha_{t,1} = t\alpha_1 + 2t^2\alpha_2 + 3t^3\alpha_3 + \cdots$. But note that we must have $\alpha_{t,1} = \delta_t\psi_1$, where δ_t is the Hochschild coboundary operator of the deformed algebra A_t. To see what this implies we must consider both obstructions to derivations and the behaviour of cohomology under deformation.

4 Derivations

If ϕ is a derivation of a $\mathbb{Q}$-algebra A then $\exp t\phi = 1 + t\phi + (t^2/2!)\phi^2 + \cdots$ is a formal automorphism, but in characteristic $p > 0$ we must ask when, given ϕ, there is an automorphism Φ_t of the form $1 + t\phi + t^2\phi_2 + t^3\phi_3 + \cdots$. The corresponding identities are then

$$\sum_{i+j=n;\ i,j>0} \phi_i \smile \phi_j = -\delta\phi_n.$$

Taking, as we may, $\phi_i = \phi^i/i!$ for $i < p$, the obstruction to finding ϕ_p is the cohomology class of $-(\phi_1 \smile \phi_{p-1} + \phi_2 \smile \phi_{p-2} + \cdots + \phi_{p-1} \smile \phi_1)$ (which is guaranteed to be a cocycle) in $\mathrm{H}^2(A, A)$. If this vanishes, then the next obstruction, which is again in H^2, is at the p^2-place; if this is passed then the following is at the p^3-place, and so forth. If we have an expression of the form $1 + t^m\phi_m + t^{m+1}\phi_{m+1} + \cdots + t^n\phi_n$ which is an automorphism of $A[t]/t^{n+1}$, then ϕ_m is necessarily a derivation, and if $mp^r \leq n$ it necessarily passes at least r obstructions. While in accordance with what one expects, proofs of some of the foregoing statements are not completely obvious: *cf.* [10].

5 Effect of deformations on cohomology

If A_t is a deformation of A, then the deformed coboundary operator δ_t of A_t can be written as a power series $\delta + t\delta_1 + t^2\delta_2 + \cdots$ where each δ_i maps the n-cochains $C^n = C^n(A, A)$ linearly into the $(n+1)$-cochains. (In fact using the graded Lie multiplication on $\bigoplus C^n$ introduced in [7], if $F \in C^n$ then $\delta F = -[\alpha, F]$ and $\delta_i F$ is just $-[\alpha_i, F]$.) Suppose now for simplicity that k is a field. An n-cocycle z is *liftable* to A_t if there is a series $z_t = z + tz_1 + t^2z_2 + \cdots$ with $z_i \in C^n$ such that $\delta_t z_t = 0$; denote the space of these by LZ^n. It is *liftable modulo* t^r if $\delta_t z_t \equiv 0 \pmod{t^r}$; denote the module of these by $L_{r-1}Z^n$. Then $Z^n = L_0Z^n \supset L_1Z^n \supset \cdots \supset LZ^n \supset B^n$, the coboundaries; the last inclusion follows because every δy is liftable to $\delta_t y$. Now $\delta_t z_t = 0$ is equivalent to having $\delta z_r = -(\delta_1 z_{r-1} + \delta_2 z_{r-2} + \cdots + \delta_r z)$. Denote the right hand side by y, the *obstruction cocycle*. The obstruction to extending a lifting $z + tz_1 + \cdots + t^{r-1}z^{r-1}$ modulo t^r to one modulo t^{r+1} is the class $[y] \in \mathrm{H}^{n+1}$. Observe that if $z + tz_1 + \cdots + t^{r-1}z_{r-1}$ is a lifting modulo t^r, then $\delta_t(z + tz_1 + \cdots + t^{r-1}z_{r-1})$ has the form $t^r(\delta_1 z_{r-1} + \delta_2 z_{r-2} + \cdots + \delta_r z)$ + higher terms, so if y is the obstruction cocycle to extending some lifting mod t^r to one mod t^{r+1} then $t^r y$ can be lifted to a δ_t-coboundary (and in particular, y itself is liftable). Such y (now in Z^{n+1}) will be called *r-jump cocycles*. The space of them is denoted $J_r Z^{n+1}$, and the union by JZ^{n+1}, so we have $LZ^{n+1} \supset JZ^{n+1} \supset \cdots \supset J_r Z^{n+1} \supset J_{r-1}Z^{n+1} \supset \cdots \supset J_0 Z^{n+1} = B^{n+1}$. Now if we extend coefficients from $k[[t]]$ to its quotient field $k((t))$, then any $y \in J_r Z^{n+1}$ can already be lifted to a coboundary. It is easy to see that every element of $Z^n(A_t, A_t)$ is the lifting of an element of Z^n, so over the field $k((t))$, we have $\mathrm{H}^n_t = \mathrm{H}^n(A_t, A_t) = LZ^n/JZ^n$, the liftable cocycles modulo the jump cocycles. (An evident strengthening which we shall not need, that $L_{r-1}Z^n/L_r Z^n \cong J_r Z^{n+1}/J_{r+1}Z^{n+1}$, leads to the invariance of the Euler–Poincaré characteristic under deformation whenever it is defined.) All we shall need of this is the

Lemma 1 *If $[z] \in \mathrm{H}^n(A, A)$ is a jump class then it is the obstruction to lifting a cocycle in $Z^{n-1}(A, A)$. In particular if $\mathrm{H}^{n-1}(A, A) = 0$ then there can be no jump classes.* ∎

6 Implications of a jump

Suppose that $\alpha_t = \alpha + t^r\alpha_r + t^{r+1}\alpha_{r+1} + \cdots$ is a jump deformation of A with $[\alpha_r] \neq 0$. From §3, this implies that $rt^r\alpha_r + (r+1)t^{r+1}\alpha_{r+1} + \cdots$ becomes a coboundary in A_t, so $r\alpha_r$ is a jump cocycle, and hence so is α_r unless we are in characteristic p and $p \mid r$. In fact, it is then possible that $\alpha_{t,1} = 0$, which will be the case if α_t is actually a power series in t^p. In that case $\alpha_{(1+u)t} = \alpha_t + u^p(t^p\alpha_p + 2t^{2p}\alpha_{2p} + \cdots) + \cdots$, and Ψ_u might actually have begun with $u^s\psi_s$ for some $s > 1$. But then $\Psi_u^{-1}\alpha_t(\Psi_u \otimes \Psi_u)$ will begin $\alpha_t + u^s\delta_t\psi_s + u^{s+1}(\cdots)$, and this must agree with $\alpha_{(1+u)t}$. Suppose the r above is mp^ℓ with $p \not| m$ so $\alpha_{(1+u)t} = \alpha_t + u^{p^\ell}(mt^{mp^\ell}\alpha_{mp^\ell} + \cdots) + \cdots$. Then surely $s \leq p^\ell$, inequality being possible if $\delta_t\psi_s = 0$, and if $s = p^\ell$ then again α_{mp^ℓ} is a jump cocycle. If $s < p^\ell$ then ψ_s is a derivation of A_t which we may write as a power series in t whose first non-zero term must then be a derivation of A. Were it inner, *i.e.*, a coboundary, then it could be lifted to a coboundary of A_t, and there can be no obstruction to extending a coboundary to an automorphism—and in fact to an inner automorphism. But that would have no effect on the equivalence. Thus, either the first non-zero term of α_t is a jump cocycle or we have found an outer derivation of A. But in either case, we have

Lemma 2 *If α_t is a jump deformation of A then $\mathrm{H}^1(A, A) \neq 0$.* ∎

7 The corollary to the DF conjecture

Suppose again that G is a finite group, p a prime dividing $|G|$ and k an algebraically closed field of characteristic p. A DF deformation of kG, if one exists, is necessarily a jump deformation; the deformed algebra, being semisimple, has no cohomology in positive dimensions and therefore no further deformations. It follows now that $\mathrm{H}^1(kG, kG) \neq 0$, and therefore that $\mathrm{H}^1(\mathbb{F}_pG, \mathbb{F}_pG) \neq 0$, since the former is obtained from the latter just by extending coefficients.

We have now seen that if the DF conjecture is true then the Hochschild cohomology group $\mathrm{H}^1(\mathbb{F}_pG, \mathbb{F}_pG)$ must be non-trivial. With the following standard theorem one can translate this requirement into group cohomology with trivial coefficients.

Theorem 3 *([1] 2.11.2) There is an isomorphism of additive groups*

$$\mathrm{H}^n(\mathbb{F}_pG, \mathbb{F}_pG) \cong \bigoplus \mathrm{H}^n(C_G(g), \mathbb{F}_p),$$

where the sum is over a set of representatives g of conjugacy classes in G . ■

Therefore there must exist a g in G for which $\mathrm{H}^1(C_G(g), \mathbb{F}_p)$ is non-trivial. But this is just the set of group homomorphisms from $C_G(g)$ to the additive group of $\mathbb{F}_p$, and there is a non-trivial homomorphism if and only if $C_G(g)$ has a normal subgroup of index p. Therefore we have

Theorem 4 *If the DF conjecture holds for a group G then there is a $g \in G$ whose centralizer has a normal subgroup of index p.* ■

Note that our consequence of the DF conjecture is in a sense dual to Cayley's theorem that G contains an element of order p.

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University of Pennsylvania, Philadelphia PA 19104–6395
mgersten@pennsas.upenn.edu
Institut für Experimentelle Mathematik, Ellernstrasse 29,
D–45326 Essen, Germany.
david@exp-math.uni-essen.de