The 3-local cohomology of the Mathieu group M_{24}

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Introduction. In this paper we calculate the localisation at the prime 3 of the integral cohomology ring of the Mathieu group M_{24}, together with its mod-3 cohomology ring. The main results are:

THEOREM 1. *The ring* $\mathrm{H}^*(M_{24}, \mathbf{Z})_{(3)}$ *is the commutative graded* $\mathbf{Z}_{(3)}$*-algebra with generators*

Generator	β	θ	ν	ξ
Degree	4	16	11	12
Additive order	3	3	3	3^2

and relations $\nu^2 = 0$ *and* $\beta\theta = 0$. *The Chern classes of the Todd representation in* $GL_{11}\,\mathbf{F}_2$ *generate the even-degree part of this ring.*

THEOREM 2. *The commutative graded* $\mathbf{F}_3$*-algebra* $\mathrm{H}^*(M_{24}, \mathbf{F}_3)$ *has generators*

Generator	B	b	N	n, X	x	T	t
Degree	3	4	10	11	12	15	16

and relations

$$Bn = bN \qquad TX = Tn = tN \qquad tX = tn$$
$$n^2 = B^2 = T^2 = N^2 = bt = bT = nN = tB = BN = BT = NT = 0$$
$$bX = nX = BX = NX = X^2 = 0.$$

In [**9**], Thomas uses our results to prove that the elliptic cohomology of the classifying space BM_{24} is generated by Chern classes, and is therefore concentrated in even dimensions.

1. The Mathieu group M_{24}**.** The Mathieu group M_{24} is a 5-transitive degree 24 permutation group of order $2^{10}.3^3.5.7.11.23$. We can read off the 3-local structure we require from the Atlas [**2**]. The Sylow 3-subgroups are isomorphic to 3^{1+2}_+, the extraspecial 3-group of order 3^3 and exponent 3. This has a presentation

$$3^{1+2}_+ \cong \langle A,\, B,\, C \mid A^3 = B^3 = C^3 = 1,\, CA = AC,\, CB = BC,\, AB = BAC\rangle.$$

Let P be a Sylow 3-subgroup of G. We see that each 3^2 is self-centralising, and that the Sylow 3-normaliser $N = N_G(P)$ is isomorphic to $3^{1+2}_+{:}\, D_8$. The outer automorphism group of 3^{1+2}_+ is isomorphic to $GL_2\mathbf{F}_3$, which has Sylow 2-subgroups isomorphic to the semidihedral group SD_{16}. As SD_{16} has exactly one subgroup isomorphic to D_8, there is only one conjugacy class of subgroups of $GL_2\mathbf{F}_3$ isomorphic to D_8. Hence, choosing new generators for P if necessary, we may assume that the D_8 is generated by elements J and K as follows: conjugation by J sends A to B^2, sends B to A and fixes C; and conjugation by K sends A to B^2, sends B to A^2 and C to C^2.

There are two conjugacy classes of elements of order 3 in M_{24}. We may assume that we have chosen generators for P and N/P such that in P, the elements of class 3A are C^r, A^rC^t and B^rC^t, whereas $A^rB^rC^t$ and $A^rB^{-r}C^t$ have class 3B. Here $r \in \{1, 2\}$ and $t \in \{0, 1, 2\}$.

2. The 3-local integral cohomology. We shall now calculate the 3-local integral cohomology ring, using a well-known result from the book of Cartan and Eilenberg.

THEOREM 3. ([**1**]) *Let G be a finite group with Sylow p-subgroup P. Recall that a class x in $\mathrm{H}^*(P, \mathbf{Z})_{(p)}$ is* stable *if, for each g in G, the image (under conjugation by g) of x in $\mathrm{H}^*(P^g, \mathbf{Z})_{(p)}$ has the same restriction to $P \cap P^g$ as has x itself.*

The restriction map from G to P is an isomorphism between $\mathrm{H}^*(G, \mathbf{Z})_{(p)}$ and the ring of stable classes in $\mathrm{H}^*(P, \mathbf{Z})_{(p)}$.

Here P is 3^{1+2}_+, whose integral cohomology was calculated by Lewis.

THEOREM 4. ([**6**]) *The cohomology ring $\mathrm{H}^*(3^{1+2}_+, \mathbf{Z})$ is generated by*

Generator	$\alpha_1,\ \alpha_2$	$\nu_1,\ \nu_2$	κ	ζ
Degree	2	3	4	6
Additive order	3	3	3	3^2

.

The ν_i square to zero. The remaining relations are:

$$\begin{array}{llll} \alpha_i\kappa = -\alpha_i^3 & \alpha_1\nu_2 = \alpha_2\nu_1 & \alpha_1\alpha_2^3 = \alpha_1^3\alpha_2 & \\ \nu_i\kappa = -\alpha_i^2\nu_i & \kappa^2 = \alpha_1^4 - \alpha_1^2\alpha_2^2 + \alpha_2^4 & \alpha_2^3\nu_1 = \alpha_1^3\nu_2 & \nu_1\nu_2 = \pm 3\zeta. \end{array}$$

The automorphism which sends A to $A^{r'}B^{s'}C^{t'}$, B to $A^rB^sC^t$ and C to C^j fixes κ, sends ζ to $j^3\zeta$ and sends

$$\alpha_1 \mapsto r'\alpha_1 + r\alpha_2 \qquad \alpha_2 \mapsto s'\alpha_1 + s\alpha_2 \qquad \nu_1 \mapsto j(r'\nu_1 + r\nu_2) \qquad \nu_2 \mapsto j(s'\nu_1 + s\nu_2).$$

■

We start by calculating the cohomology of N: this is the ring of classes in $\mathrm{H}^*(P, \mathbf{Z})_{(3)}$ which are invariant under the action of the Sylow 3-normaliser, *i.e.*, under conjugation by J and K.

PROPOSITION 5. *The ring $\mathrm{H}^*(N, \mathbf{Z})_{(3)}$ is generated by $\alpha = \alpha_1^2 + \alpha_2^2$, κ, $\eta = \zeta^2$ and $\nu = (\alpha_1\nu_1 + \alpha_2\nu_2)\zeta$. Additive exponents are obvious, and ν squares to zero. The other relation is $\alpha^2 = \kappa^2$.*

Proof. We wish to diagonalise the action of J. Write $\mathcal{H}_3$ for the module generated by the α_j and the ν_j over the ring generated by the α_j. Then $\mathcal{H}_3$ is an $\mathbf{F}_3$-vector space, and additively a direct summand of $\mathrm{H}^*(P, \mathbf{Z})_{(3)}$. Extending the scalars to $\mathbf{F}_9$ makes the action of J diagonalisable. Write i for a primitive fourth root of unity in $\mathbf{F}_9$.

J fixes κ and ζ, multiplies $\alpha_1 - i\alpha_2$ by i, and $\alpha_2 + i\alpha_2$ by $-i$. Hence in even degree the fixed classes are generated by κ, ζ, $\alpha_1^2 + \alpha_2^2$ and $(\alpha_1 \mp i\alpha_2)^4$. In both cases this last expression is $\alpha_1^4 + \alpha_2^4$, which is $-\kappa\alpha$. Similarily, the only odd-degree generator needed is $\alpha_1\nu_1 + \alpha_2\nu_2$, which we call μ.

K fixes κ and α, and multiplies ζ and μ by -1, whence the result. ∎

We now obtain a lower bound for the even-degree cohomology of G: in fact this bound is attained.

Proposition 6. *The Chern subring of G contains $\beta = \alpha + \kappa$, $\xi = \eta - \kappa^3$ and $\theta = (\alpha - \kappa)\eta$.*

Proof. Consider the Todd representation of G in $GL_{11}\,\mathbf{F}_2$. After lifting to characteristic zero (see [**8**], [**4**]), we obtain a generalised character χ_τ with partial character table

	1A	3A	3B
χ_τ	11	2	-1

.

The irreducible representations of 3_+^{1+2} are ρ^{xy} for $0 \le x, y \le 2$, and ρ^z for $1 \le z \le 2$. They have characters

$$\begin{aligned} \chi^{xy}\colon A^r B^s C^t &\longmapsto \omega^{rx+sy} \\ \chi^{z}\colon A^r B^s C^t &\longmapsto \begin{cases} 3\omega^{zt} & r = s = 0 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where ω is of course $\exp\{2\pi i/3\}$. We have $\chi_\tau = \chi^{00} + \chi^{10} + \chi^{20} + \chi^{01} + \chi^{02} + \chi^1 + \chi^2$. Let ρ_τ be a virtual representation affording χ_τ.

Theorem 7. ([**5**]) *These irreducible representations of 3_+^{1+2} have total Chern classes $c(\rho^{xy}) = 1 + x\alpha_1 + y\alpha_2$ and $c(\rho^z) = 1 + \kappa + z^3\zeta$.* ∎

Using the Whitney sum formula,

$$\begin{aligned} c(\rho_\tau) &= (1-\alpha_1^2)(1-\alpha_2^2)(1-\kappa+\kappa^2-\zeta^2) \\ &= 1 - (\alpha+\kappa) - (\eta-\kappa^3) + (\alpha\eta - \alpha\kappa^3 - \kappa^4) + (\alpha\kappa+\kappa^2)\eta. \end{aligned}$$

So $c_2(\rho_\tau) = -\beta$, $c_6 = -\xi$, and $c_8 = -(\theta + \beta\xi + \beta^4)$. ∎

In general, $\mathrm{H}^*(N, \mathbf{Z})_{(3)}$ need not be closed under the action on $\mathrm{H}^*(P, \mathbf{Z})_{(3)}$ of an automorphism of P. However, in the proof of Theorem 1 we will need to be able to approximate any automorphism of P by one that does act on $\mathrm{H}^*(N, \mathbf{Z})_{(3)}$.

Lemma 8. *Let ϕ be an automorphism of P, and D a non-central element of P. Then there is an automorphism ψ of P such that ψ equals ϕ on $\langle D, C\rangle$, and also $\mathrm{H}^*(N, \mathbf{Z})_{(3)}$ is closed under the action of ψ on $\mathrm{H}^*(P, \mathbf{Z})_{(3)}$. The map ψ^* fixes κ and η, and multiplies α and ν by ϵ, where ϵ is $+1$ or -1 according as D and ϕD are in the same or different conjugacy classes of G.*

Proof. The automorphism group of P acts transitively on the non-central elements, and hence transitively on the subgroups of order 3^2. Therefore it suffices to prove the lemma for $D = B$.

Let ϕB be $A^r B^s C^t$, and let ϕC be C^j. We shall find a and b such that defining ψA to be $A^a B^b$ gives us an automorphism ψ with the required properties. For ψ to be well-defined, we need $j \equiv as - rb$. Now, ψ^* sends $\alpha = \alpha_1^2 + \alpha_2^2$ to $(a^2+b^2)\alpha_1^2 - (ar+bs)\alpha_1\alpha_2 + (r^2+s^2)\alpha_2^2$. There is a unique solution modulo 3 to the equations $as - rb \equiv j$ and $ar + bs \equiv 0$. This also satisfies $a^2 + b^2 = r^2 + s^2$. Hence $\psi^*\alpha$ is in $\mathrm{H}^*(N, \mathbf{Z})_{(3)}$, and $\psi^*\nu$ is too. Finally, κ and η are fixed by all automorphisms of P, and r^2+s^2 is $+1$ or -1 according as ϕB is in 3A or 3B. ∎

Proposition 9. ([**6**]) *Let D be $A^r B^s C^t$. Then the ring $\mathrm{H}^*(C_3^D \times C_3^C, \mathbf{Z})_{(3)}$ is generated by δ and γ in degree 2, and χ in degree 3. All three generators have additive order 3, and χ squares to zero. The automorphism of $C_3 \times C_3$ which switches the two factors sends $\delta \leftrightarrow \gamma$ and $\chi \mapsto -\chi$. Restriction from P sends α_1 to $r\delta$, ν_1 to $r\chi$, α_2 to $s\delta$, ν_2 to $s\chi$, κ to $-\delta^2$ and ζ to $\gamma^3 - \gamma\delta^2$.* ∎

Proof of Theorem 1. We have to obtain the stable classes in $\mathrm{H}^*(P, \mathbf{Z})_{(3)}$. In Proposition 5 we calculated $\mathrm{H}^*(N, \mathbf{Z})_{(3)}$, which consists of those classes which are stable with respect to each g in $N_G(P)$. We now have to consider each g which is not in $N_G(P)$. We can ignore those P^g whose intersection with P has order 3, because corestriction from C_3 to $C_3 \times C_3$ is zero. (See Proposition 18 of [**3**].)

So we may suppose that $P^g \cap P$ has order 3^2. Such g do exist, because G contains $3^2 : GL_2\mathbf{F}_3$. The groups 3^2 in G contain either two or eight elements of class 3A, and the centre of a Sylow 3-subgroup contains two elements of 3A. Now P and P^g cannot have the same centre, for both would have to lie in the centraliser in G of a 3A: this is the triple cover $\hat{3}.A_6$, but A_6 is T.I. at 3. Hence $P \cap P^g$ contains eight elements of class 3A, and is therefore $\langle A, C\rangle$ or $\langle B, C\rangle$.

Suppose that $P \cap P^g$ is $\langle D, C\rangle$, with $D = A^r B^s C^t$ central in P^g. So D is in 3A. Lemma 8 allows us to construct an automorphism ψ of P with $\psi D = gCg^{-1}$ and $\psi C = gDg^{-1}$, such that ψ^* fixes every element of $\mathrm{H}^*(N, \mathbf{Z})_{(3)}$. Let f be the automorphism of $\langle D, C\rangle$ which switches the two factors around.

Including $\langle D, C\rangle$ in P^g and then conjugating by g is the same map to P as applying f, then including in P and then applying ψ. So a class x in $\mathrm{H}^*(N, \mathbf{Z})_{(3)}$ is in $\mathrm{H}^*(G, \mathbf{Z})_{(3)}$ if and only if its restriction to $\langle D, C\rangle$ is fixed by f^*.

Since $r^2 + s^2 \equiv 1 \pmod 3$, restriction sends α to δ^2, ν to $\delta\gamma(\gamma^2 - \delta^2)\chi$, κ to $-\delta^2$ and η to $\gamma^2(\gamma^2 - \delta^2)^2$. We immediately see that ν is stable, and generates the odd-degree stable classes over the even-degree stable classes. We know from Proposition 6 that $\alpha+\kappa$, $\eta-\kappa^3$ and $(\alpha-\kappa)\eta$ are stable, and we can now easily verify this. We claim that these three classes generated the even-degree stable classes. Since $(\alpha+\kappa)(\eta-\kappa^3)-(\alpha+\kappa)^4=(\alpha+\kappa)\eta$, they certainly generate $\kappa\eta$.

Let x be a (homogeneous) stable class of even degree. Subtracting powers of $\eta-\kappa^3$ if necessary, x contains no lone powers of η (*i.e.*, x involves no momonial of the form η^ℓ). Since $(\alpha+\kappa)^{t+1}=(-1)^t\kappa^t(\alpha+\kappa)$, we may further assume that x contains no lone powers of κ. Then x cannot contain a lone $\alpha\kappa^t$, because the restriction of x would contain a lone power of δ without the corresponding power of γ required for being fixed by f^*. Hence every term in x is divisible by $\alpha\eta$ or $\kappa\eta$. Since $\alpha^2=\kappa^2$, the only terms not divisible by $\kappa\eta$ are of the form $\alpha\eta^{t+1}$, which can be eradicated by subtracting $(\alpha-\kappa)\eta(\eta-\kappa^3)^t$. So x can be reduced to $\kappa\eta x'$. Then x' is stable, and x is a polynomial in our supposed generators if x' is. Since x' has lower degree, the claim follows by induction. Finally, the relations are obvious. ∎

3. The mod-3 cohomology. Recall that to the short exact sequence

$$0 \to \mathbf{Z}_{(3)} \xrightarrow{3\times} \mathbf{Z}_{(3)} \xrightarrow{j} \mathbf{F}_3 \to 0 \tag{1}$$

of coefficient modules there is an associated long exact sequence

$$\cdots \xrightarrow{\partial} \mathrm{H}^n(G,\mathbf{Z})_{(3)} \xrightarrow{3\times} \mathrm{H}^n(G,\mathbf{Z})_{(3)} \xrightarrow{j_*} \mathrm{H}^n(G,\mathbf{F}_3) \xrightarrow{\partial} \mathrm{H}^{n+1}(G,\mathbf{Z})_{(3)} \xrightarrow{3\times} \cdots \tag{2}$$

of cohomology groups. Using the properties of this long exact sequence, we shall derive the structure of $\mathrm{H}^*(M_{24},\mathbf{F}_3)$ from that of $\mathrm{H}^*(M_{24},\mathbf{Z})_{(3)}$.

Recall that the Bockstein homomorphism $\Delta=j_*\circ\partial$ is a graded derivation, and that the connecting map ∂ has a property akin to Frobenius reciprocity: if $x\in\mathrm{H}^n(G,\mathbf{F}_3)$ and $y\in\mathrm{H}^m(G,\mathbf{Z})_{(3)}$, then $\partial(xj_*(y))=\partial(x)y$.

First we derive the Poincaré series of $\mathrm{H}^*(M_{24},\mathbf{F}_3)$:

Theorem 10. *The $\mathbf{F}_3$-cohomology ring of M_{24} has Poincaré series*

$$\frac{1+t^3+t^4+t^7+t^8+t^{10}+3t^{11}+t^{12}+t^{14}+3t^{15}+t^{16}+t^{18}+t^{19}+t^{22}+t^{23}+t^{26}}{(1-t^{12})(1-t^{16})}.$$

Proof. Consider the long exact sequence (2) of cohomology groups. Each non-zero monomial in the generators of Theorem 1, lying in $\mathrm{H}^n(G,\mathbf{Z})_{(3)}$, contributes one basis vector to $\mathrm{H}^{n-1}(G,\mathbf{F}_3)$, and one to $\mathrm{H}^n(G,\mathbf{F}_3)$. This does apply to the ξ^ℓ, but naturally not to 1. So we calculate the generating function $f(t)$ for the number of non-zero monomials which lie in $\mathrm{H}^n(G,\mathbf{Z})_{(3)}$.

If the only generator were β, then $f(t)$ would be $1/(1-t^4)$; if θ were the only generator, it would be $1/(1-t^{16})$. Since $\beta\theta=0$, the generating function for the subring they together generate is

$$\frac{1}{1-t^4}-1+\frac{1}{1-t^{16}}=\frac{1+t^4+t^8+t^{12}+t^{16}}{1-t^{16}}.$$

The subrings generated by ν and by ξ have generating functions $1+t^{11}$ and $1/(1-t^{12})$ respectively. Since we have already budgeted for all the relations, we have

$$f(t)=\frac{1+t^4+t^8+t^{12}+t^{16}}{1-t^{16}}\times(1+t^{11})\times\frac{1}{1-t^{12}}.$$

By the argument at the start of this proof, the desired Poincaré series is then $f(t)+(f(t)-1)/t$. ∎

Proof of Theorem 2. We use the cohomology long exact sequence (2) associated to the short exact sequence (1) of coefficient modules. Define $b=j_*(\beta)$, $t=j_*(\theta)$, $n=j_*(\nu)$ and $x=j_*(\xi)$. By exactness there are unique $B\in \mathrm{H}^3(G,\mathbf{F}_3)$ and $N\in \mathrm{H}^{10}$ such that $\partial(B)=\beta$ and $\partial(N)=\nu$. We want T such that $\partial(T)=\theta$. This only defines T up to adding a multiple of bn. Since BT is in H^{18}, which has basis bnB, there is a unique T in H^{15} satisfying $\partial(T)=\theta$ and $BT=0$. There is similarly a unique $\bar{X}$ in H^{11} defined by $\partial(\bar{X})=3\xi$ and $B\bar{X}=0$. We shall set $X=\pm\bar{X}$, with the sign to be determined later. Since the image of ∂ is the ideal in $\mathrm{H}^*(G,\mathbf{Z})_{(3)}$ generated by β, θ, ν and 3ξ, we have a complete set of generators.

Most relations follow immediately. To prove that bT is zero, note that it lies in H^{19}, which is an $\mathbf{F}_3$-vector space with basis b^2n, b^4B, bxB. Applying ∂ demonstrates that bT is a scalar multiple of b^2n. Multiplying by B then shows that bT is zero, for $\partial(Bb^2n)=\beta^3\nu$, which is non-zero.

Since N^2 lies in H^{20}, it is a linear combination of b^5 and b^2x. Multiplication by n shows that N^2 is zero, since nN is zero for degree reasons.

To prove that $TX=Tn$ and $tX=tn$, we need a more intricate argument. Since TX lies in H^{26}, it must be an $\mathbf{F}_3$-linear combination of Bb^3n, Bxn and Tn. Since $bT=0$, multiplication by b shows that TX must be a scalar multiple of Tn. Applying the Bockstein map, tX is the same multiple of tn. Since we can choose $X=\pm\bar{X}$, it is enough to prove that $t\bar{X}\neq 0$.

Let $D=A^rB^sC^t$ be a non-central element of P. When we restrict from G to $\langle D,C\rangle$, by Proposition 9 we have

$$\operatorname{Res}\beta=(r^2+s^2-1)\delta^2 \qquad \operatorname{Res}\theta=(r^2+s^2+1)\gamma^2\delta^2(\gamma^2-\delta^2)^2. \tag{3}$$

Observe that $r^2+s^2\equiv 1 \pmod 3$ if D is of class 3A, and -1 if D is 3B. We have $t=\begin{cases}-j_*(\gamma^2\delta^2(\gamma^2-\delta^2)^2) & D\in 3\mathrm{A}\\ 0 & D\in 3\mathrm{B}\end{cases}$. Hence if $D\in 3\mathrm{A}$, $\operatorname{Res}t$ is neither zero nor a zero divisor. For if $y\in \mathrm{H}^m(\langle D,C\rangle,\mathbf{F}_3)$ with $m>0$ and $ty=0$, then $\partial(ty)=0$, and so $\gamma^2\delta^2(\gamma^2-\delta^2)^2\,\partial(y)=0$. If follows quickly from Proposition 9 that $\partial(y)=0$, and so $y=j_*(\upsilon)$ for some $\upsilon\in \mathrm{H}^m(\langle D,C\rangle,\mathbf{Z})_{(3)}$. Since j_* is an injection here, it follows from $ty=0$ that $\gamma^2\delta^2(\gamma^2-\delta^2)^2\upsilon=0$, whence $\upsilon=0$ and $y=0$. So it is enough to prove that, for some $D\in 3\mathrm{A}$, $\operatorname{Res}\bar{X}\neq 0$.

Similarily, $b=\begin{cases}0 & D\in 3\mathrm{A}\\ j_*(\delta^2) & D\in 3\mathrm{B}\end{cases}$. So, if $D\in 3\mathrm{B}$, then, as above, $\operatorname{Res}b$ is neither zero nor a zero divisor. But $b\bar{X}=0$, and so if $D\in 3\mathrm{B}$ then $\operatorname{Res}\bar{X}=0$.

A result of Milgram and Tezuka [**7**] states that the maximal elementary abelian subgroups of 3^{1+2}_{+} detect every non-zero element of $\mathrm{H}^*(3^{1+2}_{+}, \mathbf{F}_3)$. Hence, for some $D \in 3\mathrm{A}$, $\mathrm{Res}\,\bar{X} \neq 0$. Note that in the special case of $\bar{X}$, Milgram and Tezuka's result can be quickly verified. For $\bar{X} \in \mathrm{H}^{11}(P, \mathbf{F}_3)$ is non-zero and in the kernel of Δ, and so is a non-zero $\mathbf{F}_3$-linear combination of the images under j_* of $\alpha_1^4\nu_1$, $\alpha_1^3\alpha_2\nu_1$, $\alpha_1^2\alpha_2^2\nu_1$, $\alpha_2^4\nu_2$, $\alpha_1\nu_1\zeta$, $\alpha_2\nu_1\zeta$ and $\alpha_2\nu_2\zeta$. But we can quickly check from Proposition 9 that any non-zero $\mathbf{F}_3$-linear combination of these elements is detected by restriction to the four maximal elementary abelian subgroups.

We have now established the claimed relations. These show us that, as a module over the ring generated by x and $b^4 + t$, $\mathrm{H}^*(G, \mathbf{F}_3)$ is generated as a module by the twenty elements 1, B, b, bB, b^2, b^2B, b^3, b^3B, T, t, N, n, nB, bn, bnB, b^2n, b^2nB, b^3n, b^3nB, X. As the free module with these generators has the correct Poincaré series, there are no further relations. ∎

Remark. The author is grateful to the referee for the observation that the ring $\mathrm{H}^*(Aut(M_{12}), \mathbf{F}_3)$ is isomorphic to $\mathrm{H}^*(M_{24}, \mathbf{F}_3)$. For we see from the Atlas [**2**] that M_{24} has a maximal subgroup isomorphic to $Aut(M_{12})$, and that this contains copies of both $3^{1+2}_{+}{:}\,D_8$ and $3^2 : GL_2\mathbf{F}_3$. Consequently, we may apply the proof of Theorem 1 to $Aut(M_{12})$ and deduce that restriction from $\mathrm{H}^*(M_{24}, \mathbf{Z})_{(3)}$ to $\mathrm{H}^*(Aut(M_{12}), \mathbf{Z})_{(3)}$ is a ring isomorphism. Now, the only information about M_{24} that we use in calculating $\mathrm{H}^*(M_{24}, \mathbf{F}_3)$ is the structure of the ring $\mathrm{H}^*(M_{24}, \mathbf{Z})_{(3)}$ and the fact that the Sylow 3-subgroups of M_{24} are isomorphic to 3^{1+2}_{+}. It therefore follows that $\mathrm{H}^*(Aut(M_{12}), \mathbf{F}_3)$ is isomorphic to $\mathrm{H}^*(M_{24}, \mathbf{F}_3)$.

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