# Constructing of solutions of the sloshing problem in spherical domain

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# Boundary value problem

$$L_m\varphi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial z^2} - \frac{m^2}{r^2}\varphi = 0 \text{ in } G,\ \frac{\partial\varphi}{\partial n} = 0 \text{ on } L,$$

$$\frac{\partial\varphi}{\partial z} = \lambda\varphi \text{ on } \Gamma, \quad m = 0, 1, 2, ..., \int_0^{r_0} r\varphi dr = 0 \text{ when } m = 0. \quad (1)$$

where

$$G = \{(r,z) : r^2 + (z+s+1)^2 - 1 < 0, r > 0, z \leq -s+h-2\},$$

$$L = \{(r,z) : r^2 + (z+s+1)^2 - 1 = 0, r > 0, z \leq -s+h-2\},$$

$$\Gamma = \{(r,z) : 0 < r < r_0 = \sqrt{2h-h^2}, z = -s+h-2\}.$$

**Stokes-Zhukovsky' potential.**

$$L_1\varphi^* = 0 \text{ в } G, \quad \frac{\partial\varphi^*}{\partial n} = r\cos(n,z) - z\cos(n,r) \text{ на } L+\Gamma. \qquad (2)$$

# Inversion transform.

We change the variables $\xi$ i $\eta$.

$$r = \frac{a^2\xi}{P^2}, \quad z = \frac{a^2\eta}{P^2}, \quad \xi = \frac{a^2 r}{R^2}, \quad \eta = \frac{a^2 z}{R^2}, \tag{3}$$

where $R^2 = r^2 + z^2, \quad P^2 = \xi^2 + \eta^2, \quad PR = a^2.$

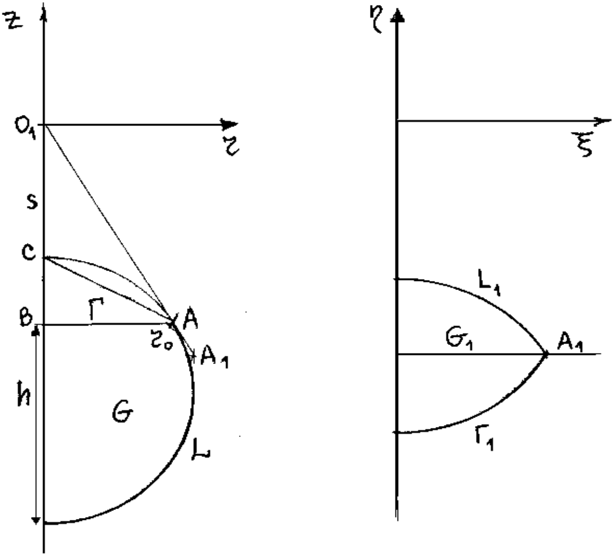
$$L_1 = \{(\xi,\eta) : \xi^2 + \left(\eta + \frac{a^2(s+1)}{s^2+2s}\right)^2 - \frac{a^4}{(s^2+2s)^2}) = 0\},$$

$$\Gamma_1 = \{(\xi,\eta) : \xi^2 + \left(\eta + \frac{a^2}{2(s+2-h)}\right)^2 - \frac{a^4}{4(s+2-h)^2}) = 0\},$$

We choose $s$ from condition of equality of both circles

$$\frac{a^2}{2s+s^2} = \frac{a^2}{2(s+2-h)}, \ 2s + s^2 = 2(s+2-h), \ s = \sqrt{4-2h},$$

$$h = 2 - s^2/2, \ r_0 = s\sqrt{1 - s^2/4}.$$


z
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$$AB^2 = r_0^2 = 1 - (h-1)^2 = 2h - h^2,\ BC = 2 - h,$$

$$AC^2 = AB^2 + BC^2 = 2h - h^2 + 4 - 4h + h^2 = 4 - 2h = s^2.$$

We choose $a$ from the condition of mapping point $A$ into $A_1$, where $r = 1$

$$O_1A \cdot O_1A_1 = a^2,\ O_1A^2 = AB^2 + (s + s^2/2)^2 = 2s^2 + s^3,$$

$$O_1A_1 = \frac{O_1A}{r_0}, a^2 = \frac{O_1A^2}{r_0} = \frac{2s^2 + s^3}{s\sqrt{1 - s^2/4}} = \frac{2s^2 + 4s}{\sqrt{4 - s^2}}.$$

$$L_1 = \{(\xi, \eta) : \xi^2 + \left(\eta + \frac{2(s+1)}{\sqrt{4 - s^2}}\right)^2 - \frac{4}{4 - s^2} = 0\},$$

$$\Gamma_1 = \{(\xi, \eta) : \xi^2 + \left(\eta + \frac{2}{\sqrt{4 - s^2}}\right)^2 - \frac{4}{4 - s^2} = 0\}.$$

If $\varphi$ is a solution of equation

$$L_m\varphi(r,z) = 0 \tag{4}$$

in domain $G$ then function $\psi(r,z) = \frac{a}{R}\varphi(\frac{a^2r}{R^2}, \frac{a^2z}{R^2})$ is a solution of this equation in domain $G_1$. The following inverse equality holds true

$$\varphi(r,z) = \frac{a}{R}\psi(\frac{a^2r}{R^2}, \frac{a^2z}{R^2}). \tag{5}$$

Let us look for boundary conditions for the function $\psi(r,z)$.

$$\frac{\partial\varphi(r,z)}{\partial n} = \frac{a}{R}\frac{\partial\psi(\frac{a^2r}{R^2}, \frac{a^2z}{R^2})}{\partial n} -$$

$$-\frac{a}{R^3}(z\cos(n,z) + r\cos(n,r))\psi(\frac{a^2r}{R^2}, \frac{a^2z}{R^2}).$$

An infinitesimal $\triangle n$ transforms into an infinitesimal $\triangle \nu$ after inversion transform and the following correspondence holds true

$$|\triangle n| = \frac{a^2|\triangle \nu|}{P^2},$$

from where we have

$$\frac{\partial \psi(\frac{a^2 r}{R^2}, \frac{a^2 z}{R^2})}{\partial n} \, |_L = \frac{P^2}{a^2} \frac{\partial \psi(\xi, \eta)}{\partial \nu} \, |_{L_1} \; .$$

$$\cos(n,r)\mid_L = r,\ \cos(n,z)\mid_L = z+s+1,\ \cos(n,r)\mid_\Gamma = 0,\ \cos(n,z)\mid_\Gamma = 1.$$

$$\frac{\partial\psi}{\partial\nu} + \sigma_1\psi = 0 \text{ on } L_1, \quad \text{where } \sigma_1 = \frac{(s+1)\eta}{P^2} + \sqrt{1-\frac{s^2}{4}}. \qquad (6)$$

$$\frac{\partial\psi}{\partial\nu} + \sigma_2\psi = \mu\sigma_3\psi \text{ on } \Gamma_1, \quad \text{where } \sigma_2 = \frac{1}{2}\sqrt{1-\frac{s^2}{4}}, \qquad (7)$$

$$\sigma_3 = -\frac{1}{\eta}\sqrt{\frac{2+s}{2-s}},\ \mu = \lambda r_0.$$

$$L_m\psi = 0 \text{ in } G_1,\ \frac{\partial\psi}{\partial\nu} + \sigma_1\psi = 0 \text{ on } L_1,\ \frac{\partial\psi}{\partial\nu} + \sigma_2\psi = \mu\sigma_3\psi \text{ on } \Gamma_1. \qquad (8)$$

Let function $\phi^*(r, z)$ corresponds to function

$$\psi^*(r, z) = \frac{a}{R}\phi^*(\frac{a^2 r}{R^2}, \frac{a^2 z}{R^2}),$$

which is a solution of the following boundary value problem

$$L_1\psi^* = 0 \text{ в } G_1, \quad \frac{\partial\psi^*}{\partial\nu} + \sigma_2\psi = \frac{a^5 r}{R^5} \text{ на } \Gamma_1, \tag{9}$$
$$\frac{\partial\psi^*}{\partial\nu} + \sigma_1\psi = \frac{a^5 r(s+1)}{R^5} \text{ на } L_1.$$

We construct solutions of problems (8) and (9) using variational method by means of minimization of appropriate functionals. For the problem (8) functional has a form

$$F(u) = \int_{G_1} \left( \xi \left( \frac{\partial \psi}{\partial \xi} \right)^2 + \xi \left( \frac{\partial \psi}{\partial \eta} \right)^2 + \frac{m^2}{\xi} \psi \right) dG + \qquad (10)$$

$$+ \int_{\Gamma_1} \xi \sigma_2 \psi^2 dl + \int_{L_1} \xi \sigma_3 \psi^2 dl,$$

$$\int_{\Gamma_1} \xi \sigma_3 \psi^2 dl = 1, \quad \psi \in W^1_{2,r}(G).$$

It has a form on the class of solutions of equation (4)

$$F(u) = \int_{L_1 + \Gamma_1} \xi \frac{\partial \psi}{\partial \nu} \psi dl + \int_{\Gamma_1} \xi \sigma_2 \psi^2 dl + \int_{L_1} \xi \sigma_1 \psi^2 dl \qquad (11)$$

$$\int_{\Gamma_1} \xi \sigma_3 \psi_i \psi dl = 0, \ (i = 1, 2, ... n - 1). \qquad (12)$$

Solution of the problem (9) minimizes the following functional

$$F(u) = \int_{L_1+\Gamma_1} \xi \frac{\partial \psi}{\partial \nu} \psi dl + \int_{\Gamma_1} (\xi \sigma_2 \psi^2 - 2 f_1 \psi) dl + \qquad (13)$$

$$+ \int_{L_1} (\xi \sigma_1 \psi^2 - 2 f_2 \psi) dl, \; f_1 = \frac{a^5 r}{R^5}, \; f_2 = \frac{a^5 r}{R^5}(s+1),$$

on the class of solutions of equation $L_1 \psi = 0$.

We use Ritz method for minimization functionals (11) and (13) by means of approximation solutions by sums $\psi_N = \sum_{i=1}^N a_i w_i$. As a result, problem (8) is reduced to generalized spectral problem of linear algebra

$$(A - \lambda B)X = 0. \text{ where} \tag{14}$$

$$\alpha_{k,j} = \int_{L_1} \xi \left( \frac{\partial w_k}{\partial \nu} + \sigma_1 w_k \right) w_j dl + \int_{\Gamma_1} \xi \left( \frac{\partial w_k}{\partial \nu} + \sigma_2 w_k \right) w_j dl,$$

$$\beta_{k,j} = r_0 \int\limits_{\Sigma} \xi \sigma_3 w_k w_j dS.$$

We have the following system of linear algebraic equations for constants $a_i$, $\sum_{j=1}^N \alpha_{i,j} a_j = c_i$, where $c_i = \int_{\Gamma_1} w_i f_1 dl + \int_{L_1} w_i f_2 dl$.

Efficiency of variational method depends on system of coordinate functions. We construct in this work solutions of equation (4) with singularities except traditional homogeneous harmonic polynomials

$$w_k(r,z) = \frac{2^m m!(k-m)!}{\pi m! i^m} \int_0^{\pi} (z + ir\cos(t))^k \cos(mt)dt \qquad (15)$$

$$(k = m, m+1, ...).$$

It is known that isolated point of bounded harmonic function is removable singularity. That is why if singularity is not removable and situated in the corner point of the domain then it is not isolated. This confirms necessity of including to coordinate functions that themselves or their derivative have discontinuity out of the domain.

# Solutions of equation (4)

Let function $f(z)$ is twice continuously differentiable. Then function

$$v(r,z)=\int_0^{\pi} f(u+rcos(t))\cos(mt)dt \tag{16}$$

satisfies equation

$$B_m v \equiv \frac{\partial^2 v}{\partial u^2}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{m^2}{r^2}\psi_m=0 \tag{17}$$

Let us consider such functions and recurrent equalities for them

$$v_k(r,z)=\int_0^{\pi}(u+rcos(t))^{k-0.5}\cos(mt)dt. \tag{18}$$

$$\frac{\partial v_k(r,u)}{\partial u}=(k-0.5)\int_0^{\pi}(u+rcos(t))^{k-1.5}cos(mt)dt= \tag{19}$$

$$=(k-0.5)v_{k-1}(r,u).$$

$$r\frac{\partial v_k(r,u)}{\partial r} = r(k-0.5)\int_0^{\pi}(u+rcos(t))^{k-1.5}\cos(mt)cos(t)dt =$$
$$(k-0.5)(\int_0^{\pi}(u+rcos(t))^{k-0.5}\cos(mt)dt - \qquad (20)$$
$$-u\int_0^{\pi}(u+rcos(t))^{k-1.5}\cos(mt)dt) =$$
$$(k-0.5)(v_k(r,u) - uv_{k-1}(r,u).$$

$$((k+\frac{1}{2})^2 - m^2)v_{k+1}(r,u) = ((2k+1)kuv_k(r,u) +$$
$$+(r^2-u^2)(k^2-\frac{1}{4})v_{k-1}(r,u). \qquad (21)$$

We obtain the following solutions of equation (4) based on constructed functions:

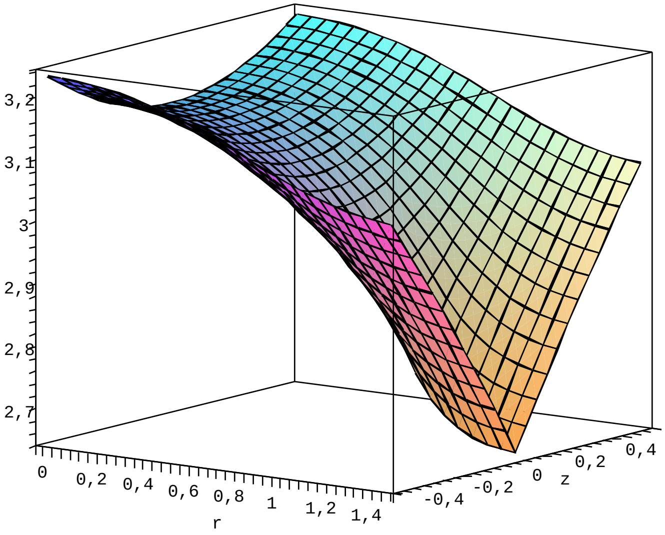
$$w_{2k-1}^{*}(r,z) = Im(v_k(r, iz + r_0)), \quad w_{2k}^{*}(r,z) = Re(v_k(r, iz + r_0)), \tag{22}$$

$$\psi_m^N(r,z) = \sum_{k=1}^{N} a_k w_k(r,z) + \sum_{k=1}^{N_1} a_{k+N} w_k^{*}(r,z). \tag{23}$$

$$v_1(r,u) = 2\sqrt{u-r}E(\sqrt{\frac{2r}{r-u}}) \text{ when } m = 0.$$

$$v_1(r,u) = \frac{2\sqrt{u-r}}{3r}\left(uE(\sqrt{\frac{2r}{r-u}}) - (u+r)K(\sqrt{\frac{2r}{r-u}})\right) \text{ when } m = 1.$$

Here $E$ and $K$ are entire elliptic integrals of the first and the second order. The other functions can be represented via entire elliptic integrals too. Further there are graphs of the first two functions.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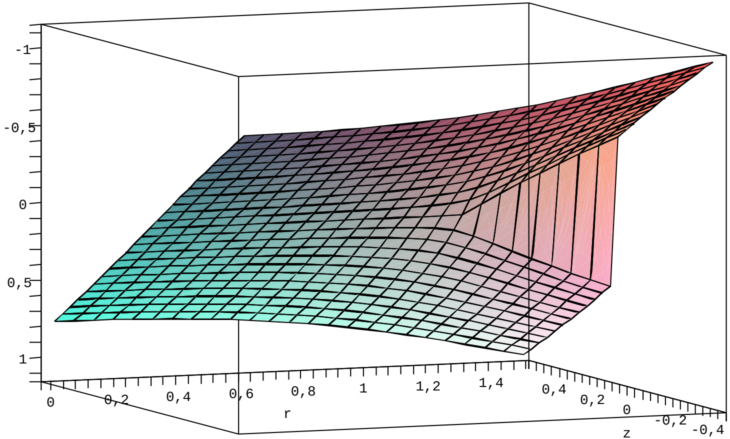
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r
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The tables below have results of eigenvalues calculations of problem (8) (right column) for different heights of liquid filled spherical cavity in depending on the number of recorded functions $w_k^*(r, z)$ from 0 to 6 (first column). In the other columns it is shown accuracy of boundary conditions close to the corner point on $L$ and i $\Gamma$. Boundary conditions are fulfilled the worst in the neighbor of a corner point, due to the presence singularities. Incorporation functions with singularities provides an opportunity to get mo precise solutions.

Table 1.($h = 1.0$, $n = 20$)

| $n_1$ | $\delta_\Gamma(0.01)$ | $\delta_\Gamma(0.02)$ | $\delta_L(0.01)$ | $\delta_L(0.02)$ | $\mu_k$ |
|---|---|---|---|---|---|
| 0 | -0.00689 | 0.00071 | 0.00686 | -0.00086 | 1.56015720702 |
| 1 | -0.00218 | -0.00101 | 0.00213 | 0.00087 | 1.56015717750 |
| 2 | -0.00210 | -0.00095 | 0.00220 | 0.00092 | 1.56015717749 |
| 4 | 0.00000 | -0.00001 | -0.00001 | 0.00000 | 1.56015717383 |
| 6 | 0.00000 | -0.00000 | -0.00000 | 0.00000 | 1.56015717382 |
| 0 | -0.02006 | 0.00105 | 0.01960 | -0.00370 | 5.27554633367 |
| 1 | -0.00679 | -0.00396 | 0.00600 | 0.00156 | 5.27554608302 |
| 2 | -0.00548 | -0.00305 | 0.00731 | 0.00248 | 5.27554608045 |
| 4 | 0.00019 | 0.00012 | 0.00014 | 0.00016 | 5.27554604679 |
| 6 | 0.00001 | 0.00000 | -0.00001 | 0.00003 | 5.27554604673 |
| 0 | -0.03252 | -0.00016 | 0.03088 | -0.00832 | 8.50444184552 |
| 1 | -0.01175 | -0.00827 | 0.00921 | 0.00046 | 8.50444119136 |
| 2 | -0.00746 | -0.00539 | 0.01350 | 0.00343 | 8.50444116390 |
| 4 | 0.00063 | 0.00047 | 0.00059 | 0.00057 | 8.50444106584 |
| 6 | 0.00004 | -0.00009 | -0.00022 | 0.00016 | 8.50444106497 |

Table 2.($h = 1.5$, $n = 20$)

| $n_1$ | $\delta_\Gamma(0.01)$ | $\delta_\Gamma(0.02)$ | $\delta_L(0.01)$ | $\delta_L(0.02)$ | $\mu_k$ |
|---|---|---|---|---|---|
| 0 | -0.10674 | -0.04765 | 0.10910 | 0.04659 | 2.04577026375 |
| 1 | -0.00026 | -0.01188 | 0.00192 | 0.01042 | 2.04576434634 |
| 2 | 0.00011 | -0.01058 | 0.00228 | 0.01173 | 2.04576434040 |
| 4 | -0.00306 | -0.00046 | 0.00297 | 0.00060 | 2.04576409158 |
| 6 | -0.00062 | 0.00049 | 0.00067 | -0.00048 | 2.04576408694 |
| 0 | -0.35089 | -0.16582 | 0.38165 | 0.15170 | 5.51929009634 |
| 1 | 0.00744 | -0.04881 | 0.01534 | 0.03015 | 5.51922071278 |
| 2 | 0.01313 | -0.03148 | 0.02045 | 0.04773 | 5.51921963615 |
| 4 | -0.01065 | -0.00027 | 0.00846 | 0.00283 | 5.51921619451 |
| 6 | -0.00150 | 0.00145 | 0.00247 | -0.00181 | 5.51921614252 |
| 0 | -0.64764 | -0.32253 | 0.74897 | 0.27542 | 8.72493550656 |
| 1 | 0.02827 | -0.10819 | 0.04755 | 0.04661 | 8.72468001146 |
| 2 | 0.04953 | -0.05019 | 0.06569 | 0.10592 | 8.72466782939 |
| 4 | -0.02230 | 0.00203 | 0.01314 | 0.00777 | 8.72465223123 |
| 6 | -0.00057 | 0.00111 | 0.00620 | -0.00492 | 8.72465196499 |

Table 3.($h = 2.0$, $n = 20$)

| $n_1$ | $\delta_\Gamma(0.01)$ | $\delta_\Gamma(0.02)$ | $\delta_L(0.01)$ | $\delta_L(0.02)$ | $\mu_k$ |
|---|---|---|---|---|---|
| 0 | 1.12055 | 0.96048 | -1.26542 | -1.05760 | 2.75754081328 |
| 1 | 0.08874 | 0.04671 | 0.05107 | 0.02980 | 2.75477140300 |
| 2 | 0.02685 | 0.02590 | -0.01467 | 0.00672 | 2.75475603699 |
| 4 | -0.00479 | -0.00392 | -0.00591 | -0.00367 | 2.75475477570 |
| 6 | -0.00132 | -0.00017 | -0.00135 | -0.00008 | 2.75475474333 |
| 0 | 2.28461 | 1.98527 | -2.85011 | -2.37308 | 5.90628138899 |
| 1 | 0.40819 | 0.20607 | 0.12932 | 0.07845 | 5.89242015081 |
| 2 | 0.18171 | 0.13624 | -0.12998 | -0.01077 | 5.89218405668 |
| 4 | -0.01434 | -0.01835 | -0.03257 | -0.01515 | 5.89214828759 |
| 6 | -0.00464 | -0.00017 | -0.00498 | -0.00011 | 5.89214748459 |
| 0 | -3.40712 | -2.99557 | 4.54942 | 3.80190 | 9.07102231000 |
| 1 | 0.96321 | 0.47018 | 0.09148 | 0.05974 | 9.03409329400 |
| 2 | 0.54383 | 0.35602 | -0.42962 | -0.11502 | 9.03314967857 |
| 4 | 0.01135 | 0.05035 | 0.10353 | 0.03535 | 9.03286132085 |
| 6 | -0.00482 | 0.00304 | -0.00595 | 0.00041 | 9.03285271615 |

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