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Epigraph

Theory without practice cannot survive and dies as quickly as it lives

Leonardo da Vinci 1452-1519, cited from M.Kline, Math. Thought 1972, p. 224

Theory and practice in the DFG team with A. Timokha (Trondheim, Norwey), M.Barnyak, I. Lukovsky, A.Solodun, V.Vasylyk, V., V.Trotsenko and Yu. Trotsenko (Kyiv, Ukraine), M.Kutniv (Lviv, Ukraine)

Some mathematical problems of hydrodynamics

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1. Dezember 2009

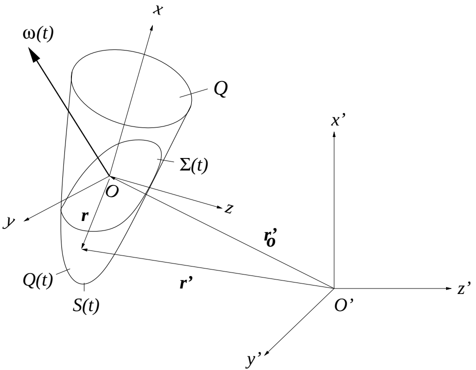
(a)Boynton Beach, Florida (8 Megaliter) (b)Sydney (3.2 Megaliter)


Moss type
LNG Carriers
Membrane type
LNG Carriers

Open-top fire in a naphtha tank caused by earthquake-induced sloshing

Oiltanker - USA, Texas, Galveston 1990: Norwey oilltanker "Mega Borg" in Golf of Mexiko, 90 km from the coast. 14.000 t crude oil build ca. 75 km long oil film

Sloshing in a tank



Sloshing in a tank

$\zeta(x, y, z, t) = 0$ - equation of free surface
$\phi(x, t)$ -the velocity potential (a harmonic function of $x = (x_1, x_2, x_3) \in \tilde{\Omega}$

$$\Delta\phi = 0 \quad \text{in} \quad Q,$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on} \quad S, \; \frac{\partial\phi}{\partial n} = -\frac{\zeta_t}{\sqrt{(\nabla\zeta)^2}} \quad \text{on} \quad \Sigma \quad \text{(kinematic b. c.)},$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gx = 0 \quad \text{on} \quad \Sigma \quad \text{(dynamic b.c.)}$$

Sloshing in a tank

$\zeta(x,y,z,t) = x - f(y,z,t) \Rightarrow \frac{\partial\phi}{\partial x} = \frac{\partial f}{\partial t}$ - simplified kinematic b.c.
Differentiated (in t) dynamic b.c.+simplified kinematic b.c. $\Rightarrow$

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial x} = 0$$

Sloshing in a tank

A new unknown function u: the velocity potential ϕ on the unperturbed free surface Σ_0

These functions are connected by the equations

$$\Delta\phi = 0 \text{ in } Q,$$

$$\phi|_{\Sigma_0} = u, \quad \phi_n|_S = 0$$

The function ϕ depends on u uniquely and linearly so that in a space H of functions depending on x, y a linear operator $\mathcal{A}$ can be defined (the Dirichlet-to-Neumann operator)

$$\mathcal{A}u = \left.\frac{\partial\phi}{\partial x}\right|_{\Sigma_0}$$

Linear sloshing in a tank: problem in the time domain

Now the boundary condition on Σ_0 yields the equation

$$\frac{d^2u}{dt^2} + \mathcal{A}u = 0 \text{ on } \Sigma_0,$$
$$u(0) = u_0, \quad u'(0) = 0,$$

where $u : \mathbb{R}_+ \to H$ is a vector-valued function
The solution operator is the operator cosine function
$C(t) = \cos\left(\sqrt{A}t\right)$:

$$u(t) = C(t)u_0$$

Linear sloshing in a chute: problem in the frequency domain

Eigenoscillations: $u = e^{i\lambda t} v, \quad v \neq v(t)$

$$-\lambda^2 v + \mathcal{A} v = 0$$

Equivalent formulation provided that
$\phi(x, y, z, t) = \cos(\sigma t + \epsilon)\tilde{\phi}(x, y, z, t),\ \kappa = \frac{\sigma^2}{g}$:

$$\Delta\tilde{\phi} = 0 \text{ in } Q,$$
$$\frac{\partial\tilde{\phi}}{\partial x} = \kappa\tilde{\phi} \quad \text{on} \quad \Sigma_0, \quad \frac{\partial\tilde{\phi}}{\partial n} = 0 \quad \text{on} \quad S$$

Nonlinear sloshing in a tank: modal models

Variational problem:

$$W = -\rho \int_{t_1}^{t_2} \int_Q \left[\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + gx \right] dQ$$

Ansatz:

$$x = f(y,z,t) = \sum_{i=1}^{\infty} \beta_i(t) f_i(y,z), \quad \phi(x,y,z,t) = \sum_{n=1}^{\infty} R_n(t) \phi_n(x,y,z)$$

Result: an initial or boundary value problem for a system of nonlinear ODE w.r.t. $\beta_i(t)$, $R_n(t)$

$n-$width: a measure of accuracy

$$X \in C(\overline{D}); \phi : X \to \mathbf{R}^n, f \to \xi = (f(t_1), ..., f(t_n))$$

$$d[\phi^{-1}(\xi)] = \sup_{g,h \in \phi^{-1}(\xi)} |g - h|_\infty$$

- a measure which shows the goodness of the approximation of an element $f|\phi(f) = \xi$ by ξ

$$\mathcal{E}(X, \mathbf{R^n}; \phi) = \sup_{f \in X} d[\phi^{-1} \circ \phi(f)]$$

- the accuracy of the approximation in worth case

$n-$width: a measure of accuracy

$$\Delta_n(X, C(\overline{D})) = inf_\phi \mathcal{E}(X, \mathbf{R^n}; \phi)$$

- the grid $n-$width: the measure of goodness of the best possible approximation of an arbitrary element of X by its values on the grid

Curse of dimensionality

Let $X = W_p^r(M; I)$ with $M = (M_1, ..., M_d)$ and $r = (r_1, ..., r_d)$ be the class of anisotropic Sobolev spaces defined on the d-dimensional interval $I = \prod_{j=1}^{d}[a_j, b_j]$.
$\rho = 1/(\sum_{j=1}^{d} r_j^{-1})-$ the effective *class smoothness*
$\mu = \prod_{j=1}^{d} M_j^{\rho/r_j}-$ the *class constant*.
It is known that for this class we need

$$N_\varepsilon^{(opt)} \asymp const(\mu) \cdot \varepsilon^{-1/\,(\rho - 1/p)}$$

parameters (optimal complexity) in order to approximate an arbitrary function of this class with a given tolerance ε. Note that $N_\varepsilon^{(opt)}$ grows exponentially as $d \to \infty$ ("curse of dimensionality"). $|\log \varepsilon| = \log \frac{1}{\varepsilon}$ is the measure unity for the best possible

Curse of dimensionality for analytical functions

$$N_\varepsilon = \mathcal{O}\left((\log|\log\varepsilon|)^d |\log\varepsilon|^d \right)$$

numbers are needed to approximate an analytical function of d variables, i.e. $|\log\varepsilon| = \log\frac{1}{\varepsilon}$ remains the measure unity for the best possible approximation

$\Rightarrow$ an exponential approximation accuracy required!

Sol. for Curse of d. and exp. conv. alg.

$$u'(t) + Au = 0, \quad u(0) = u_0$$

$$u(t) = e^{-At} u_0 = \int_\Gamma e^{-zt} (z - A)^{-1} dz \; u_0$$

$Sinc - quadrature \;\Rightarrow$ exp. conv.+ linear in d algorithms!

EDS for BVPs for systems of nonlinear ODEs

$$\mathbf{u}'(x) + A(x)\mathbf{u} = \mathbf{f}(x, \mathbf{u}), \quad x \in (0,1), \quad B_0\mathbf{u}(0) + B_1\mathbf{u}(1) = \mathbf{d}, \ (1)$$

$$A(x), B_0, B_1, \in \mathbb{R}^{d\times d}, \quad \mathrm{rank}[B_0, B_1] = d, \quad \mathbf{f}(x, \mathbf{u}), \mathbf{d}, \mathbf{u}(x) \in \mathbb{R}^d$$

$\mathbf{u}$ is an unknown $d-$dimensional vector-function

$$\hat{\bar{\omega}}_h = \{x_j : \ 0 = x_0 < x_1 < x_2 < \cdots < x_N = 1\}$$

EDS for BVPs for systems of nonlinear ODEs

IVPs (each of the dimension d)

$$\frac{d\mathbf{Y}^j(x,\mathbf{v}_{j-1})}{dx} + A(x)\mathbf{Y}^j(x,\mathbf{v}_{j-1}) = \mathbf{f}(x,\mathbf{Y}^j(x,\mathbf{v}_{j-1})), \quad x \in (x_{j-1},x_j],$$

$$\mathbf{Y}^j(x_{j-1},\mathbf{v}_{j-1}) = \mathbf{v}_{j-1}, \quad j = 1,2,\ldots,N. \tag{2}$$

- multishooting method

EDS for BVPs for systems of nonlinear ODEs

$$\mathbf{u}_j = \mathbf{Y}^j(x_j, \mathbf{u}_{j-1}), \quad j = 1, 2, \dots N, \tag{3}$$

$$B_0 \mathbf{u}_0 + B_1 \mathbf{u}_N = \mathbf{d}. \tag{4}$$

Truncated difference scheme

m-TDS:

$$\mathbf{y}_j^{(m)} = Y^{(m)j}(x_j, \mathbf{y}_{j-1}^{(m)}), \quad j = 1, 2, \dots N, \tag{5}$$

$$B_0 \mathbf{y}_0^{(m)} + B_1 \mathbf{y}_N^{(m)} = \mathbf{d} \tag{6}$$

$Y^{(m)j}(x_j, \mathbf{y}_{j-1}^{(m)})$ - the num. sol. of IVP (2) on $[x_{j-1}, x_j]$ which has been obtained by some one-step method of the order m

$$\left\| \mathbf{y}^{(m,n)} - \mathbf{u} \right\|_{0,\infty,\widehat{\overline{\omega}}_h} \leq M \left(q_2^n + |h|^m \right), \tag{7}$$

where $q_2 \equiv q + M\,|h| < 1$, n is the number of the fixed point iteration step

A posteriori error estimation and automatic grid generation is based on the embedded IVP-solver of the orders m and $m + 1$

Trancated difference scheme: Troesch test

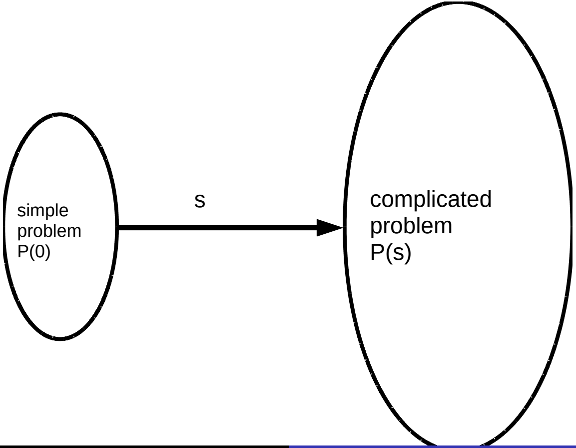
$$u'' = \lambda \sinh(\lambda u), \quad x \in (0,1), \quad \lambda > 0, \quad u(0) = 0, \quad u(1) = 1. \tag{8}$$

$\lambda = 10$ ($\lambda = 61, 62$ is possible)

ε	N	$NFUN$	$Error$
10^{-4}	72	11323	$0.143 \cdot 10^{-5}$
10^{-6}	124	19435	$0.538 \cdot 10^{-7}$

Tabelle: Numerical results for the TDS with $m = 8(7)$

The homotopy idea



FD-method: eigenvalue problem

Problem $P(t)$:

$$(A+W(t))u(t)-\lambda(t)u(t)=\theta, \quad t\in[0,1]$$

Problem $P(1)$:

$$(A+B)u-\lambda u=\theta \quad \text{in} \quad H$$

Problem $P(0)$:

$$(A+\overline{B})u^{(0)}-\lambda^{(0)}u^{(0)}=\theta$$

FD-method: eigenvalue problem

Representation:
$\lambda(t) = \sum_{j=0}^{\infty} \lambda^{(j)} t^j$, $u(t) = \sum_{j=0}^{\infty} u^{(j)} t^j$
Recurrence equations:

$$(A + \overline{B})u^{(j+1)} - \lambda^{(0)} u^{(j+1)} = F^{(j+1)},\ j = 0, 1, \dots$$

with $\lambda^{(0)},\ u^{(0)}$ - solution of $P(0)$ and

$$F^{(j+1)} = F^{(j+1)}(\lambda^{(0)}, \dots, \lambda^{(j+1)}; u^{(0)}, \dots, u^{(j)}) =$$

$$= \lambda^{(j+1)} u^{(0)} - [B - \overline{B}]u^{(j)} + \sum_{p=1}^{j} \lambda^{(j+1-p)} u^{(p)},\ j = 0, 1, \dots$$

FD-method: eigenvalue problem

Approximation for $P(1)$:

$$\overset{N}{\lambda} = \sum_{j=0}^{N} \lambda^{(j)}, \quad \overset{N}{u} = \sum_{j=0}^{N} u^{(j)}$$

Convergence result:

$$|\lambda - \overset{N}{\lambda}| \leq \|B - \overline{B}\| \frac{C}{(1+N)^{1+\varepsilon}} q^N,$$

$$\|u - \overset{N}{u}\| \leq \frac{C}{(1+N)^{1+\varepsilon}} q^{N+1}$$

provided that $q = C' \frac{\|B-\overline{B}\|}{\ldots} < 1$

FD-method: eigenvalue problem

Example:

$$u^{(4)}(x) + \left[\left(x - \frac{1}{2} \right)^2 - \lambda \right] u(x), \quad x \in (0,1)$$
$$u^{(k)}(0) = u^{(k)}(1) = 0, \qquad k = 2, 3.$$

Here the operators A, B are defined by
$D(A) = \{v \in H^4(0,1) : v^{(k)}(0) = v^{(k)}(1) = 0,\ k = 2, 3\},\ Av = v^{(4)}(x)\ \forall v \in D(A); D(B) = L_2(0,1),\ Bv = \left(x - \frac{1}{2}\right)^2 v(x).$

FD-method: Example

$$u^{(4)}(x) + \left[\left(x - \frac{1}{2} \right)^2 - \lambda \right] u(x), \quad x \in (0,1)$$
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FD-method: Example

The pair of smallest eigenvalues is
$\lambda_{0,-}^{ex} = 0.0833223112249938...,\ \lambda_{0,+}^{ex} = 0.14999891773580...$
The base problem $P(0)$ with $\overline{B} = 0$ possesses the double eigenvalue $\lambda^{(0)} = 0$ corresponding to the orthonormal eigenfunctions
$u_{01}(x) = 1,\ u_{02}(x) = 12x - 6.$

Rank N	$\overset{N}{\lambda}_{0,-}(\overline{B})$	$\overset{N}{\lambda}_{0,+}(\overline{B})$
0	0	0
1	0.08(3)	0.15
2	0.0833223104056437	0.14999891774891
3	0.0833223112249098	0.14999891773580

FD-method: Example

The error:

Rank N	$\lvert\lambda_{0,-}^{ex} - \overset{N}{\lambda}_{0,-}(\overline{B})\rvert$	$\lvert\lambda_{0,+}^{ex} - \overset{N}{\lambda}_{0,+}(\overline{B})\rvert$
0	$0.84 \cdot 10^{-1}$	0.15
1	$0.12 \cdot 10^{-4}$	$0.11 \cdot 10^{-5}$
2	$0.83 \cdot 10^{-9}$	$0.14 \cdot 10^{-10}$
3	$0.84 \cdot 10^{-13}$	$0.39 \cdot 10^{-15}$

The IVP

$$\frac{d^2u(t)}{dt^2} + Au(t) = f(t), \quad u(0) = u^{(0)}, \quad u'(0) = u^{(1)}$$

The operator cosine and sine families

$$B \equiv 0, \quad u^{(1)} \equiv 0 \Rightarrow u(t) = C(t)u^{(0)}$$

$$\frac{d^2 C(t)}{dt^2} + AC(t) = 0, \quad C(0) = I, \quad C'(0) = 0$$

$$S(t) = S(t; \sqrt{A}) = \int_0^t C(t)dt$$

Representation of solution

$$u(t) = u_h(t) + u_p(t)$$

$$u_h(t) = C(t)u^{(0)} + A^{-1/2}S(t)u^{(1)}$$

$$= \cos(\sqrt{A}t)u^{(0)} + A^{-1/2}\sin(\sqrt{A}t)u^{(1)}$$

$$u_p(t) = \int_0^t S(t-s)f(s)ds$$

The Cayley transform method for cos

$$
\begin{aligned}
x(t) \equiv x(t;A) &= (\cos\sqrt{A}t)x_0 \\
&= \frac{1}{2}e^{-\delta t}\sum_{n=0}^{\infty}\left(L_n^{(0)}(t) - L_{n-1}^{(0)}(t)\right) \times (W_+^n + W_-^n)x_0 \\
&= e^{-\delta t}\sum_{n=0}^{\infty}\left(L_n^{(0)}(t) - L_{n-1}^{(0)}(t)\right) u_n,
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
C(t) \equiv \cos\sqrt{A}t &= \frac{1}{2}e^{-\delta t}\sum_{n=0}^{\infty}\left(L_n^{(0)}(t) - L_{n-1}^{(0)}(t)\right)(W_+^n + W_-^n) \\
&= e^{-\delta t}\sum \left(L_n^{(0)}(t) - L_{n-1}^{(0)}(t)\right)\mathcal{U}_n,
\end{aligned}
$$

The Cayley transform method for cos

The sequences of vectors $\{u_n\}$ and of operators $\{\mathcal{U}_n\} \equiv \{\mathcal{U}_n(A)\}$ are defined by

$$\begin{aligned} u_{n+1} &= 2(A+\delta(\delta-1)I)(A+(\delta-1)^2 I)^{-1} u_n \\ &\quad - (A+\delta^2 I)(A+(\delta-1)^2 I)^{-1} u_{n-1} \quad , \ n \geq 1, \\ u_0 &= x_0 \quad , \ u_1 = (A+\delta(\delta-1)I)(A+(\delta-1)^2 I)^{-1} x_0 \end{aligned} \tag{10}$$

and

$$\begin{aligned} \mathcal{U}_{n+1} &= 2(A+\delta(\delta-1)I)(A+(\delta-1)^2 I)^{-1} \mathcal{U}_n \\ &\quad - (A+\delta^2 I)(A+(\delta-1)^2 I)^{-1} \mathcal{U}_{n-1} \quad , \ n \geq 1, \\ \mathcal{U}_0 &= I \quad , \ \mathcal{U}_1 = (A+\delta(\delta-1)I)(A+(\delta-1)^2 I)^{-1} \end{aligned} \tag{11}$$

without using $\sqrt{A}$!

The Cayley transform method for cos

One can observe that In order to find $\{u_n\}$ one has to solve the following sequence of operator equations

$$(A+(\delta-1)^2 I)u_{n+1} = \bar{u}_{n+1} \quad , \quad n \geq 1; \ (A+(\delta-1)I)u_1 = \bar{u}_1 \tag{12}$$

with the same operator $A + (\delta - 1)^2 I$ and with various right-hand sides, where

$$\begin{aligned} \bar{u}_{n+1} &= 2(A + \delta(\delta - 1)I)u_n - (A + \delta^2 I)u_{n-1} \quad , \ n \geq 1, \\ u_0 = x_0 \quad &, \ \bar{u}_1 = (A + \delta(\delta - 1)I)u_0. \end{aligned} \tag{13}$$

The Cayley transform method for cos

Algorithm: the truncated sum with N terms
Estimate: Let A be a densely defined, strongly P-positive operator and $x_0 \in \mathcal{D}(A^\sigma)$. Then

$$\sup_{t\in[0,T]} \|x^N(t) - x(t)\| \leq cN^{-(\sigma-\frac{3}{4}-\varepsilon)}\|A^\sigma x_0\| \qquad (14)$$

No accuracy saturation, but an exponential accuracy for analytical vectors only!

The Danford-Cauchy representation

For a strongly P-positive operator:

$$C(t) = \frac{1}{2\pi i} \int_{\Gamma} \cos \sqrt{z} t (z - A)^{-1} dz$$

Parametrization+Laguerre quadrature $\Leftrightarrow$ Alg. without saturation
Other Approximations???

Tensor product representations: with MPI for Mathematics in the Sciences Leipzig

$$N_{\varepsilon}^{(opt)} = \mathcal{O}\left((\log|\log \varepsilon|)^d |\log \varepsilon|^d \right)$$

numbers are needed to represent an analytical function of d variables with a tolerance ε

$$\mathcal{L}u = f$$

in a $d-$dimensional rectangle one uses a discretization

$$\mathcal{L}_h u_h = f_h$$

with $N = n^d$ nodes and a matrix $\mathcal{L}_h \in \mathbb{R}^{n^d \times n^d}$

Tensor product representations

The complexity of computation of $u_h = \mathcal{L}_h^{-1} f_h$ is in general $\mathcal{O}(n^{2d})$ (again the curse of dimensionality)

Tensor product representations

Supposing the tensor-product representations

$$\mathcal{L}_h^{-1} = \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_d, \; f_h = f_1 \otimes f_2 \otimes \cdots \otimes f_d$$

we have due to

$$\mathcal{L}_h^{-1} f_h = (\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_d) \cdot (f_1 \otimes f_2 \otimes \cdots \otimes f_d)$$

$$= (\mathcal{L}_1 \cdot f_1) \otimes (\mathcal{L}_2 \cdot f_2) \otimes \cdots (\mathcal{L}_d \cdot f_d)$$

the polynomial in d complexity $\mathcal{O}(dn^2)$.

Epilogue

Also geht alles zu Ende allhier:
Feder, Tinte, Tobak und auch wir,
Zum letzenmal wird eingetunkt,
Dann kommt der große schwarze •
(*W. Busch, Bilder zur Jobsiade 1872*)