# Exponentially convergent method for the $m$-point problem for a first order differential equation in Banach space

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Use of exponentially convergent method and finite-difference method for the (vector) electric field in three space and one time dimension. Computer run time is shown as Days: Hours: Minutes: Seconds

| Acc. | F.D. Run-Time | Sinc-Conv. Run-Time |
|---|---|---|
| $10^{-1}$ | $\approx$ 1 second | $\approx$ 1 second |
| $10^{-2}$ | 000:00:00:27 | 000:00:00:06 |
| $10^{-3}$ | 003:00:41:40 | 000:00:02:26 |
| $10^{-4}$ | $\approx$ 82 years | 000:00:43:12 |
| $10^{-5}$ | $\approx$ 800,000 years | 000:06:42:20 |
| $10^{-6}$ | $\approx$ 8.2 billion years | 001:17:31:11 |

*F.Stenger. Summary of Sinc numerical methods. J.Comp.Appl.Math., v.121 (2000) pp. 379-420.*

$$
\begin{aligned}
&u'_t + Au = f(t), \quad t \in [0, T] \\
&u(0) + \sum_{k=1}^{m} \alpha_k u(t_k) = u_0, \quad 0 < t_1 < t_2 < \ldots < t_m \le T,
\end{aligned} \tag{1}
$$

where $\alpha_k \in \mathbb{R}$, $k = \overline{1, m}$, $f(t)$ is a given vector-valued function with values in Banach space $X$, $u_0 \in X$. The operator $A$ with the domain $D(A)$ in a Banach space $X$ is assumed to be a densely defined strongly positive (sectorial) operator, i.e. its spectrum $\Sigma(A)$ lies in a sector of the right half-plane with the vertex at the origin and and the resolvent decays inversely proportional to $|z|$ at the infinity.

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Additionally outside the sector and on its boundary $\Gamma_\Sigma$ the following estimate for the resolvent holds true

$$\left\|(zI - A)^{-1}\right\| \leq \frac{M}{1 + |z|}. \tag{2}$$

The numbers $\rho_0$, $\varphi$ are called the spectral characteristics of $A$.
The hyperbola

$$\Gamma_0 = \{z(\xi) = \rho_0 \cosh \xi - ib_0 \sinh \xi : \ \xi \in (-\infty, \infty), \ b_0 = \rho_0 \tan \varphi\} \tag{3}$$

in turn is refered as a spectral hyperbola. It has a vertex at $(\rho_0, 0)$ and asymptotes which are parallel to the rays of the spectral angle $\Sigma$.

In works of Byszewski it was proven that the solution of the problem (1) exists and is unique provided that one of the following two conditions is fulfilled:

$$\sum_{i=1}^{m} |\alpha_i| < 1, \tag{4}$$

or

$$\sum_{i=1}^{m} |\alpha_i| e^{-\rho_0 t_i} < 1, \tag{5}$$

with $\rho_0$ – spectral characteristic.

The solution of (1) is in the form $u(t) = u_h(t) + u_{ih}(t)$, where $u_h(t) = \mathrm{e}^{-At} B^{-1} u_0$ is the solution of the homogeneous problem with the initial condition $u_0$ and

$$u_{ih}(t) = -\mathrm{e}^{-At} B^{-1} \sum_{i=1}^{m} \alpha_i \int_0^{t_i} \mathrm{e}^{-A(t_i - \tau)} f(\tau) d\tau + \int_0^{t} e^{-A(t-\tau)} f(\tau) d\tau \tag{6}$$

is the solution of the inhomogeneous problem with zero initial condition and

$$B(A) = I + \sum_{i=1}^{m} \alpha_i \mathrm{e}^{-At_i}$$

First of all we consider the solution $u_h(t) = \mathrm{e}^{-At} B^{-1}(A) u_0$ of homogeneous problem (1). Our aim in this section is to construct an exponentially convergent method for its approximation.
Using the Dunford-Cauchy representation of $u_h(t)$ we obtain

$$\begin{aligned} u_h(t) &= \frac{1}{2\pi i} \int_{\Gamma_I} \mathrm{e}^{-zt} B^{-1}(z)(zI - A)^{-1} u_0 dz \\ &= \frac{1}{2\pi i} \int_{\Gamma_I} \frac{\mathrm{e}^{-zt}}{1 + \sum\limits_{i=1}^{n} \alpha_i \mathrm{e}^{-zt_i}} (zI - A)^{-1} u_0 dz. \end{aligned} \tag{7}$$

Representation (7) makes sense only when the function $\mathrm{e}^{-zt} B^{-1}(z)$ is analytic in the region enveloped by $\Gamma_I$. Condition (4) or (5) guaranty this analyticity.

Parameterizing the integral (7) we get

$$u_h(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{F}(t,\xi) d\xi, \tag{8}$$

with

$$\mathcal{F}(t,\xi) = F_A(t,\xi) u_0,$$

$$F_A(t,\xi) = \frac{\mathrm{e}^{-z(\xi)t} z'(\xi)}{1 + \sum_{i=1}^{n} \alpha_i \mathrm{e}^{-z(\xi)t_i}} \left[ (z(\xi)I - A)^{-1} - \frac{1}{z(\xi)} I \right],$$

$$z'(\xi) = a_I \sinh \xi - i b_I \cosh \xi.$$

We approximate integral (8) by the following Sinc-quadrature (F.Stenger (1993). Numerical methods based on Sinc and analytic functions. Springer Verlag. New York, Berlin, Heidelberg.):

$$u_{h,N}(t) = \frac{h}{2\pi i} \sum_{k=-N}^{N} \mathcal{F}(t, z(kh)), \tag{9}$$

with an error

$$\|\eta_N(\mathcal{F}, h)\| = \|u_h(t) - u_{h,N}(t)\|$$

$$\leq \frac{c\|A^\alpha u_0\|}{\alpha} \left\{ \frac{e^{-\pi d_1/h}}{\sinh(\pi d_1/h)} + \exp[-a_I t \cosh((N+1)h) - \alpha(N+1)h] \right\},$$

where the constant $c$ does not depend on $h, N, t$. Equalizing the both exponentials for $t = 0$ implies

$$\frac{2\pi d_1}{h} = \alpha(N+1)h,$$

or after the transformation

$$h = \sqrt{\frac{2\pi d_1}{\alpha(N+1)}}. \tag{10}$$

With this step-size the following error estimate holds true

$$\|\eta_N(\mathcal{F}, h)\| \leq \frac{c}{\alpha}\exp\left(-\sqrt{\frac{\pi d_1 \alpha}{2}(N+1)}\right)\|A^\alpha u_0\|, \tag{11}$$

with a constant $c$ independent of $t, N$. In the case $t > 0$ the first summand int the argument of $\exp[-a_I t \cosh((N+1)h) - \alpha(N+1)h]$ from the estimate for $\|\eta_N(\mathcal{F}, h)\|$ contributes mainly to the error order. Setting in this case $h = c_1 \ln N/N$ with some positive constant $c_1$ we remain, asymptotically for a fixed $t$, with an error

$$\|\eta_N(\mathcal{F}, h)\| \leq c\left[e^{-\pi d_1 N/(c_1 \ln N)} + e^{-c_1 a_I t N/2 - c_1 \alpha \ln N}\right]\|A^\alpha u_0\|, \tag{12}$$

Thus, we have proved the following result.

**Theorem (1)**

*Let $A$ be a densely defined strongly positive operator and $u_0 \in D(A^\alpha)$, $\alpha \in (0,1)$, then Sinc-quadrature (9) represents an approximate solution of the homogeneous nonlocal value problem (1) (i.e. the case when $f(t) \equiv 0$) and possesses a uniform with respect to $t \geq 0$ exponential convergence rate which is of the order $\mathcal{O}(e^{-c\sqrt{N}})$ uniformly in $t \geq 0$ provided that $h = \mathcal{O}(1/\sqrt{N})$ (estimate (11)) and of the order $\mathcal{O}\left(max\left\{\mathrm{e}^{-\pi dN/(c_1 \ln N)}, \mathrm{e}^{-c_1 a_I tN/2 - c_1 \alpha \ln N}\right\}\right)$ for each fixed $t > 0$ provided that $h = c_1 \ln N/N$ (estimate (12)).*

In this section we consider the particular solution (6) of inhomogeneous problem (1), i.e. with $f(t) \neq 0$.
Let us rewrite formula (6) in the form

$$u_{ih}(t) = u_{1,ih}(t) + u_{2,ih}(t), \tag{13}$$

with

$$u_{1,ih}(t) = \int_0^t e^{-A(t-\tau)} f(\tau) d\tau, \quad u_{2,ih}(t) = -\sum_{j=1}^{m} \alpha_j u_{2,ih,j}(t), \tag{14}$$

where

$$u_{2,ih,j}(t) = \int_0^{t_i} B^{-1} \mathrm{e}^{-A(t+t_i-\tau)} f(\tau) d\tau. \tag{15}$$

We approximate the term $u_{1,ih}(t)$ by the algorithm:

$$\begin{aligned} u_{1,ih}(t) \approx u_{1,N}(t) = & \frac{h}{2\pi i} \sum_{k=-N}^{N} z'(kh)[(z(kh)I - A)^{-1} - \frac{1}{z(kh)} I] \\ & \times h \sum_{p=-N}^{N} \mu_{k,p}(t) f(\omega_p(t)), \end{aligned} \tag{16}$$

where

$$\mu_{k,p}(t) = \frac{t}{2} \exp\{-\frac{t}{2} z(kh)[1 - \tanh(ph)]\} / \cosh^2(ph),$$

$$\omega_p(t) = \frac{t}{2}[1 + \tanh(ph)], \; h = \mathcal{O}\left(1/\sqrt{N}\right),$$

$$z(\xi) = a_I \cosh \xi - i b_I \sinh \xi, \; z'(\xi) = a_I \sinh \xi - i b_I \cosh \xi.$$

The next theorem characterizes the error of this algorithm [].

Theorem ((2), Gavrilyuk,Makarov)

*Let A be a densely defined strongly positive operator with spectral characteristics $\rho_0$, $\varphi$ and a right hand side $f(t) \in D(A^\alpha)$, $\alpha > 0$ for $t \in [0, \infty]$ can be analytically extended into the sector $\Sigma_f = \{\rho e^{i\theta_1} : \rho \in [0, \infty], |\theta_1| < \varphi\}$ where the estimate*

$$\|A^\alpha f(w)\| \le c_\alpha e^{-\delta_\alpha |Re\, w|}, \quad w \in \Sigma_f \tag{17}$$

*with $\delta_\alpha \in (0, \sqrt{2}\rho_0]$ holds, then algorithm (16) converges to the solution of (1) with the error estimate*

$$\|\mathcal{E}_N(t)\| = \|u_{1,ih}(t) - u_{1,N}(t)\| \le c e^{-c_1 \sqrt{N}} \tag{18}$$

*uniformly in t with positive constants $c$, $c_1$ depending on $\alpha$, $\varphi$, $\rho_0$ and independent of $N$.*

We propose the following algorithm to compute an approximation $u_{2,j,N}$ to $u_{2,ih,j}$

$$u_{2,ih,j}(t) \approx u_{2,j,N}(t) = \frac{h}{2\pi i} \sum_{k=-N}^{N} \mathrm{e}^{-z(kh)t} z'(kh) B^{-1}(z(kh)) \times \left[ (z(kh)I - A)^{-1} - \frac{1}{z(kh)} I \right] h \sum_{p=-N}^{N} \mu_{k,p,j} f(\omega_{p,j}). \tag{19}$$

Theorem (3)

*Let the assumptions of the theorem 2 hold. Then algorithm (19) converges uniformly with respect to t and moreover the following error estimate holds true:*

$$\|\mathcal{E}_N(t)\| = \|u_{2,ih,j}(t) - u_{2,j,N}(t)\| \leq c\mathrm{e}^{-c_1\sqrt{N}}, \tag{20}$$

*with positive constants $c$, $c_1$ depend on $\alpha$, $\varphi$, $\rho_0$ and independent of $N$.*

Theorem 3 guaranties that the error this approach will be bounded by:

$$\|u_{2,ih}(t) - u_{2,N}(t)\| \le \sum_{j=1}^{m} c\mathrm{e}^{-c_1\sqrt{N}} \le c_2\mathrm{e}^{-c_1\sqrt{N}}. \tag{21}$$

Thus, the approximations (9) together with (16) represent an exponentially convergent algorithm for the problem (1).

We consider the homogeneous problem (1) with the operator $A$ defined by

$$
\begin{aligned}
&D(A) = \{u(x) \in H^2(0,1): \; u(0) = u(1) = 0\}, \\
&Au = -u''(x) \; \forall u \in D(A).
\end{aligned}
\tag{22}
$$

The initial nonlocal condition reads as follows:

$$u(x,0) + 0.5u(x,0.2) + 0.3u(x,0.4)$$

$$= (1 + 0.5\mathrm{e}^{-\pi^2 0.2} + 0.3\mathrm{e}^{-\pi^2 0.4})\sin(\pi x),$$

with $u_0 = (1 + 0.5\mathrm{e}^{-\pi^2 0.2} + 0.3\mathrm{e}^{-\pi^2 0.4})\sin(\pi x) \in D(A)$.
The exact solution of the problem is $u(x,t) = \mathrm{e}^{-\pi^2 t}\sin(\pi x)$.

It is easy to find that

$$(zI - A)^{-1}u_0 = \left(z + \frac{d^2}{dx^2}\right)^{-1} \sin(\pi x) = \frac{\sin(\pi x)}{z - \pi^2}.$$

| N | $\varepsilon_N$ |
|---|---|
| 4 | .29857983847712589e-1 |
| 8 | .41823888073604986e-2 |
| 16 | .11258594468208641e-2 |
| 32 | .10042178166563831e-3 |
| 64 | .28007158539828452e-5 |
| 128 | .2098826601399176e-7 |
| 256 | .1858929920173152e-10 |
| 512 | .856837124351510e-15 |

Table: The error for $x = 0.5$, $t = 0.3$.

Due to Theorem 1 the error should not be greater then $\varepsilon_N = \mathcal{O}\left(\mathrm{e}^{-c\sqrt{N}}\right)$.
The constant $c$ in the exponent can be estimated using the following a-posteriori relation:

$$c = \ln\left(\frac{\varepsilon_N}{\varepsilon_{2N}}\right)(\sqrt{2}-1)^{-1}N^{-1/2} = \ln\left(\mu_N\right)(\sqrt{2}-1)^{-1}N^{-1/2}.$$

The numerical results are presented in the Table 2 for this estimation show that the constant can be estimated as $c \approx 1.5$ when $N \to \infty$.

| N | $c$ |
|---|---|
| 4 | 2.372652515388745588587496 |
| 8 | 1.120148732795449515627946 |
| 16 | 1.458741976765153165445005 |
| 32 | 1.527648924601130131250452 |
| 64 | 1.476794596387591759032900 |
| 128 | 1.499935011373075736075927 |
| 256 | 1.506597339081609844717370 |

Table: The estimate of $c$

The next example deals again with a homogeneous problem but in this more realistic case the resolvent of $A$ on the element $u_0$ can't be calculated analytically.

We consider the homogeneous problem (1) with the operator $A$ defined as in (22) and with the following initial nonlocal condition:

$$u(x,0) + u(x,0.5) = x\ln(x),$$

where $u_0 = x\ln(x) \in A^\alpha$, $\alpha < 1/2$. In this case the resolvent can be represented using the Green function

$$(zI - A)^{-1}u_0 = \left(z + \frac{d^2}{dx^2}\right)^{-1} x\ln(x) = \int_0^1 G(x,s)s\ln(s)ds,$$

$$G(x,s) = -\frac{1}{\sqrt{z}\sin(\sqrt{z})}\begin{cases}\sin(x\sqrt{z})\sin((1-s)\sqrt{z}) & x \le s,\\ \sin(s\sqrt{z})\sin((1-x)\sqrt{z}) & x \ge s\end{cases},$$

where the integrals were computed by exponentially convergent Sinc-quadrature.

| N | $u(x,t)$ |
|---|---|
| 4 | -.241535790017043e-1 |
| 8 | -.228401191029108e-1 |
| 16 | -.194273285627507e-1 |
| 32 | -.192905848633180e-1 |
| 64 | -.192911920318628e-1 |
| 128 | -.192907849909929e-1 |
| 256 | -.192907820740651e-1 |

Table: Values of the solution $u(x,t)$ for $x = 0.5,\ t = 0.3$.

It can be easily seen that the number of stabilized digits increases according to the theoretical prediction by Theorem 1.

Let us consider the inhomogeneous problem (1) with the same $A$ defined by (22), and the nonlocal condition

$$u(x,0) + 0.5u(x,0.2) = (1 + 0.5\mathrm{e}^{0.2})\sin(\pi x),$$

For $f(t,x)$ at the right hand side of the equation (1) we set

$$f(x,t) = (1+\pi^2)\mathrm{e}^t \sin(\pi x).$$

The exact solution of the problem is $u(x,t) = \mathrm{e}^t \sin(\pi x)$. We have used the algorithm defined by (9), (16).

The results presented in Table 4 and are again in good agreement with the theoretical predictions.

| N | $\varepsilon_N$ |
| --- | --- |
| 4 | .202211483120243 |
| 8 | .726677678737409e-1 |
| 16 | .138993889900620e-1 |
| 32 | .143037059411419e-2 |
| 64 | .554542099757830e-4 |
| 128 | .532640823981411e-6 |
| 256 | .730569324317506e-9 |
| 512 | .648376079810788e-13 |

Table: The error for $x = 0.5, \ t = 0.3$.

Thank you for your attention

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