Mathematical Statistics, Winter term 2018/19
Problem sheet 1

1) (i) Let

$$X = \begin{pmatrix} 1 & v_1 & v_1^2 & \cdots & v_1^k \\ 1 & v_2 & v_2^2 & \cdots & v_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & v_n & v_n^2 & \cdots & v_n^k \end{pmatrix}.$$

Prove that $X^T X$ is regular if the set $\{v_1, \ldots, v_n\}$ contains at least $k+1$ different values.
Hint: Choose $c = (c_1, \ldots, c_{k+1})^T \neq 0_{k+1} := (0, \ldots, 0)^T$ *and compute* $c^T X^T X c$.

(ii) Let

$$X = \begin{pmatrix} v_1 & v_1^2 & \cdots & v_1^k \\ v_2 & v_2^2 & \cdots & v_2^k \\ \vdots & \vdots & \ddots & \vdots \\ v_n & v_n^2 & \cdots & v_n^k \end{pmatrix}.$$

Prove that $X^T X$ is regular if the set $\{v_1, \ldots, v_n\}$ contains at least k different non-zero values.
Hint: Consider the matrix

$$\widetilde{X} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & v_1 & v_1^2 & \cdots & v_1^k \\ 1 & v_2 & v_2^2 & \cdots & v_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & v_n & v_n^2 & \cdots & v_n^k \end{pmatrix}.$$

2) Suppose that

$$Y = X\beta + \varepsilon$$

holds for some $\beta \in \mathbb{R}^k$ and that $E\varepsilon = 0_n$, $\mathrm{Cov}(\varepsilon) = \Sigma$, where Σ is a regular matrix.

(i) Show that

$$\widehat{\beta} = (X^T \Sigma^{-1} X)^{-1} X^T \Sigma^{-1} Y$$

is an unbiased estimator of β and compute $E_\beta\left[(\widehat{\beta} - \beta)(\widehat{\beta} - \beta)^T\right]$.

(ii) Let $\widetilde{\beta} = LY$ be any arbitrary unbiased estimator of β.
Show that

$$E_\beta\left[(\widetilde{\beta} - \beta)(\widetilde{\beta} - \beta)^T\right] - E_\beta\left[(\widehat{\beta} - \beta)(\widehat{\beta} - \beta)^T\right]$$

is non-negative definite.

Hint: A symmetric and positive definite $(n \times n)$-matrix M can be represented as $M = \sum_{i=1}^n \lambda_i e_i e_i^T$, where $\lambda_1, \ldots, \lambda_n$ are the (positive) eigenvalues and $e_1, \ldots, e_n$ are corresponding eigenvectors with $e_i^T e_j = 0$ for $i \neq j$. Then $M^{1/2} := \sum_{i=1}^n \sqrt{\lambda_i} e_i e_i^T$ and $M^{-1/2} := \sum_{i=1}^n (1/\sqrt{\lambda_i}) e_i e_i^T$.
To prove (ii), use the fact that

$$\left(L\Sigma^{1/2} - (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1/2}\right)\left(L\Sigma^{1/2} - (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1/2}\right)^T$$

is non-negative definite.