Mathematical Statistics, Winter term 2018/19
Problem sheet 2

3) Consider the linear regression model $Y_i = \beta_1 + x_i\beta_2 + \varepsilon_i$, $i = 1, \ldots, n$, where $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d. with $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$. Let $\widehat{\beta}$ be the least squares estimator of β.

   (i) Suppose that $x_i \neq x_j$, for some (i, j).
   Compute $E[(\widehat{\beta}_i - \beta_i)^2]$, for $i = 1, 2$.
   Hint: The inverse of a regular matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is given by* $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

   (ii) Suppose that $x_1, \ldots, x_n$ can be chosen by an experimenter, where $x_i \in [-1, 1]$ and $n \geq 2$ is even.
   Which choice of $x_1, \ldots, x_n$ minimizes $E[(\widehat{\beta}_i - \beta_i)^2]$?

4) Let X be an $(n \times k)$-matrix with $\text{rank}(X) = k$. Show that $M := X(X^TX)^{-1}X^T$ is the projection matrix onto the linear space $\mathcal{M} = \{Xb \colon\ b \in \mathbb{R}^k\}$. ($Mx = x\ \forall x \in \mathcal{M}$ and $Mx = 0_n$ if $X^Tx = 0_k$.)

5) An urn contains M red and $N - M$ black balls ($0 \leq M \leq N$). n balls are randomly chosen without replacement. The random variable X describes the number of chosen red balls.

   (i) For fixed M, what is the probability of $X = k$, for $k = 0, 1, \ldots, n$?

   (ii) Suppose that N and n are known and that M is the unknown parameter of interest. Define an appropriate statistical experiment.

   (iii) Find an estimator T of M with the property

$$E_M T = \int T(\omega)\ P_M(d\omega) = M \qquad \forall M \in \{0, 1, \ldots, N\}.$$