Mathematical Statistics, Winter term 2018/19
Problem sheet 3

6) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed, real-valued random variables with a common continuous distribution function F. Let F_n be the empiricial distribution function defined as $F_n(x) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}(X_i \le x)$.

   Show that, as $n \to \infty$,
   $$\sup_x |F_n(x) \; - \; F(x)| \longrightarrow 0$$
   holds almost surely.

   Hint: Choose, for $\varepsilon > 0$, appropriate points $-\infty = x_0 < x_1 < \cdots < x_{M-1} < x_M = \infty$ such that $F(x_i) - F(x_{i-1}) \le \varepsilon$ $(F(x_0) := 0,\, F(x_M) := 1)$ and show almost sure convergence first at these points. Then use monotonicity of F and F_n to obtain the desired result.

7) Suppose that realizations of independent and identically distributed random variables $X_1, \ldots, X_n$ are available, where $P_\theta(X_i = 1) = \theta = 1 - P_\theta(X_i = 0)$ and $\theta \in \Theta$. The performance of any estimator $T = t(X_1, \ldots, X_n)$ is measured by the quadratic risk $R(T,\theta) = E_\theta(T-\theta)^2$.

   (i) Show that there does not exist a uniformly best estimator if $\Theta = [0,1]$.

   (ii) If $\Theta = \{0,1\}$, does there exist a uniformly best estimator?

8) Let $(X, \Omega, \mathcal{A}, \{P_\theta : \theta \in \Theta\})$ be a statistical experiment. Suppose that there exists some $\theta_0 \in \Theta$ such that $P_\theta \ll P_{\theta_0}$ $\forall \theta \in \Theta$, that is, $P_{\theta_0}(B) = 0$ implies $P_\theta(B) = 0$.

   Show that the estimator $T \equiv \theta_0$ is an admissible estimator of θ under the squared error loss.

9) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with $P_\theta(X_i = 1) = \theta = 1 - P_\theta(X_i = 0)$, where $\theta \in \Theta = (0,1)$.

   Show that there is no unbiased estimator $T = t(X_1, \ldots, X_n)$ of the parameter $g(\theta) = 1/\theta$.