Mathematical Statistics, Winter term 2018/19

Problem sheet 4

10) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with $P_\theta(X_i = 1) = \theta = 1 - P_\theta(X_i = 0)$, where $\theta \in \Theta = (0, 1)$.

    Show that $t^*(X_1, \ldots, X_n) = n^{-1} \sum_{i=1}^n X_i$ is the Bayesian estimator under the loss function L with $L(t, \theta) = (t - \theta)^2 / (\theta(1 - \theta))$ and prior distribution $\pi = \text{Uniform}([0, 1])$.

    Hint: Use that $\int_0^1 \theta^k (1 - \theta)^{n-k-1} \, d\theta / \int_0^1 \theta^{k-1} (1 - \theta)^{n-k-1} \, d\theta = k/n$ *if* $0 < k < n$ *(See also the lecture on October 6.) and compute* $t^*(0, \ldots, 0)$ *and* $t^*(1, \ldots, 1)$ *separately.*

11) Let $X_1, \ldots, X_n$ be i.i.d. with $X_i \sim \mathcal{N}(\theta, \sigma^2)$. For the prior distribution $\pi = \mathcal{N}(0, \tau^2)$, the posterior distribution is given by $P^{\theta|X=x} = \mathcal{N}(\frac{n\tau^2}{n\tau^2 + \sigma^2} \bar{x}_n, \frac{\sigma^2 \tau^2}{n\tau^2 + \sigma^2})$. (See also the lecture on October 7.)

    Compute the Bayesian risk of the Bayesian estimator under the squared error loss.