**Mathematical Statistics, Winter term 2018/19**
Problem sheet 9

26) Let $X_1, \dots, X_n$ be i.i.d. with $X_i \sim \text{Uniform}([0, \theta])$, where $\theta \in (0, \infty)$.

(i) Compute the maximum likelihood estimator $\widehat{\theta}_{ML,n}$ of $\theta$.

(ii) Compute the distribution function of $n(\theta - \widehat{\theta}_{ML,n})$. What is the limit of these distribution functions as $n \to \infty$?

**Definition** A sequence of real-valued random variables $(X_n)_{n\in\mathbb{N}}$ on $(\Omega, \mathcal{A}, P)$ is *bounded in probability* (or tight), if for every $\varepsilon > 0$ there exists $M_\varepsilon < \infty$ such that

$$P\left(|X_n| > M_\varepsilon\right) \le \epsilon \quad \forall n \in \mathbb{N}.$$

27) Let $(X_n)_{n\in\mathbb{N}_0}$ be a sequence of real-valued random variables on $(\Omega, \mathcal{A}, P)$ and let $X_n \xrightarrow{d} X_0$ (convergence in distribution).

Show that $(X_n)_{n\in\mathbb{N}}$ is bounded in probability.

**Definition** Let $(X_n)_{\mathbb{N}}$ be a sequence of real-valued random variables on $(\Omega, \mathcal{A}, P)$ and let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers. Then

(i) $X_n = O_P(\alpha_n)$, if $(X_n/\alpha_n)_{n\in\mathbb{N}}$ is bounded in probability,

(ii) $X_n = o_P(\alpha_n)$, if $X_n/\alpha_n \xrightarrow{P} 0$.

28) Let $X_n = O_P(\alpha_n)$, $Y_n = O_P(\beta_n)$, and $Z_n = o_P(\beta_n)$.

Show that

(i) $X_n + Y_n \;=\; O_P(\max\{\alpha_n, \beta_n\})$,

(ii) $X_n \cdot Y_n \;=\; O_P(\alpha_n \beta_n)$,

(iii) $Z_n \;=\; O_P(\beta_n)$,

(iv) $X_n \cdot Z_n \;=\; o_P(\alpha_n \beta_n)$.