Mathematical Statistics, Winter term 2018/19

Problem sheet 10

29) Suppose that $X_1, \ldots, X_n$ are independent, with $X_i \sim \mathrm{NB}(\theta, k)$, where $\mathrm{NB}(\theta, k)$ denotes a negative binomial distribution with parameters $\theta \in \Theta := (0,1)$ and a known k, that is, $P_\theta(X_i \in \{k, k+1, \ldots\}) = 1$ and $P(X_i = m) = \binom{m-1}{k-1}\theta^k(1-\theta)^{m-k} \quad \forall m \in \{k, k+1, \ldots\}$. (If we have a sequence of independent Bernoulli trials, each of them with a success probability of θ, then the number of the kth successful trial has a negative binomial distribution with parameters θ and k.)

   (i) Compute the distribution of $T = \sum_{i=1}^n X_i$.

   (ii) Show (without using the Factorization Theorem) that T is a sufficient statistic for $\theta \in \Theta$.

30) Let $X_1, \ldots, X_n$ be i.i.d. with $X_i \sim \mathrm{Poisson}(\theta)$, where $\theta \in \Theta := (0, \infty)$.

   Show that $T = \sum_{i=1}^n X_i$ is a sufficient statistic for $\theta \in \Theta$.

31) Let $X_1, \ldots, X_n$ be i.i.d. with $X_i \sim \mathrm{Bin}(1, \theta)$, where $\theta \in \Theta := (0,1)$.

   Show (without using Theorem 1.3.5) that $\widehat{\theta} = n^{-1} \sum_{i=1}^n X_i$ is the best unbiased estimator of θ under quadratic loss.

   Hint: Prove first that $\widetilde{\theta} = Y/n$ is the unique unbiased estimator of θ if $Y \sim Bin(n, \theta)$ is observed.

32) Let $X_1, \ldots, X_n$ be i.i.d. with $EX_i = \theta$ and $EX_i^2 = 1$. Furthermore, let $X_{n:1}, \ldots, X_{n:n}$ be the order statistics, $X_\uparrow = (X_{n:1}, \ldots, X_{n:n})^T$. Let $\widehat{\theta} = \sum_{i=1}^n a_i X_i$ be an estimator of θ, where $a_1, \ldots, a_n$ are constants.

   Compute $h(x) = E(\widehat{\theta} \mid X_\uparrow = x)$ and show that the quadratic risk of $\widetilde{\theta} = h(X_\uparrow)$ is not greater than the quadratic risk of $\widehat{\theta}$.