A Appendix

A.1 Notations

Throughout the book we use the following standard notations:

1. The **natural numbers** starting at 1 are always denoted by $\mathbb{N}$. In the case 0 is included we write $\mathbb{N}_0$.
2. As usual the **integers** $\mathbb{Z}$ are given by $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
3. By $\mathbb{R}$ we denote the field of **real numbers** endowed with the usual algebraic operations and its natural order. The subset $\mathbb{Q} \subset \mathbb{R}$ is the union of all **rational numbers**, that is, of numbers m/n where $m, n \in \mathbb{Z}$ and $n \neq 0$.
4. Given $n \geq 1$ let $\mathbb{R}^n$ be the n**-dimensional Euclidean vector space**, that is,

$$\mathbb{R}^n = \{x = (x_1, \dots, x_n) : x_j \in \mathbb{R}\} .$$

Addition and scalar multiplication in $\mathbb{R}^n$ are carried out coordinate-wise,

$$x + y = (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots x_n + y_n)$$

and if $\alpha \in \mathbb{R}$, then

$$\alpha\, x = (\alpha x_1, \dots, \alpha x_n) .$$

A.2 Elements of Set Theory

Given a set M its **powerset** $\mathcal{P}(M)$ consists of all subsets of M. In the case that M is finite we have $\#(\mathcal{P}(M)) = 2^{\#(M)}$, where $\#(A)$ denotes the **cardinality** (number of elements) of a finite set A.

If A and B are subsets of M, written as $A, B \subseteq M$ or also as $A, B \in \mathcal{P}(M)$, their **union** and their **intersection** are, as usual, defined by

$$A \cup B = \{x \in M : x \in A \text{ or } x \in B\} \text{ and } A \cap B = \{x \in M : x \in A \text{ and } x \in B\} .$$

Of course, it always holds that

$$A \cap B \subseteq A \subseteq A \cup B \quad \text{and} \quad A \cap B \subseteq B \subseteq A \cup B .$$

In the same way, given subsets $A_1, A_2, \dots$ of M their union $\bigcup_{j=1}^{\infty} A_j$ and their intersection $\bigcap_{j=1}^{\infty} A_j$ is the set of those $x \in M$ that belong to at least one of the A_j or that belong to all A_j, respectively.

Quite often we use the **distributive law** for intersection and union. This asserts

$$A \cap \left(\bigcup_{j=1}^{\infty} B_j \right) = \bigcup_{j=1}^{\infty} (A \cap B_j)\,.$$

Two sets A and B are said to be **disjoint**[1] provided that $A \cap B = \emptyset$. A sequence of sets $A_1, A_2, \ldots$ is called disjoint[2] whenever $A_i \cap A_j = \emptyset$ if $i \neq j$.

An element $x \in M$ belongs to the **set difference** $A \backslash B$ provided that $x \in A$ but $x \notin B$. Using the notion of the **complementary set** $B^c := \{x \in M : x \notin B\}$, the set difference may also be written as

$$A \backslash B = A \cap B^c\,.$$

Another useful identity is

$$A \backslash B = A \backslash (A \cap B)\,.$$

Conversely, the complementary set may be represented as the set difference $B^c = M \backslash B$. We still mention the obvious $(B^c)^c = B$.

Finally we introduce the **symmetric difference** $A \Delta B$ of two sets A and B as

$$A \Delta B := (A \backslash B) \cup (B \backslash A) = (A \cap B^c) \cup (B \cap A^c) = (A \cup B) \backslash (A \cap B)\,.$$

Note that an element $x \in M$ belongs to $A \Delta B$ if and only if x belongs exactly to one of the sets A or B.

De Morgan's rules are very important and assert the following:

$$\left(\bigcup_{j=1}^{\infty} A_j \right)^c = \bigcap_{j=1}^{\infty} A_j^c \quad \text{and} \quad \left(\bigcap_{j=1}^{\infty} A_j \right)^c = \bigcup_{j=1}^{\infty} A_j^c\,.$$

Given sets $A_1, \ldots, A_n$ their **Cartesian product** $A_1 \times \cdots \times A_n$ is defined by

$$A_1 \times \cdots \times A_n := \{(a_1, \ldots, a_n) : a_j \in A_j\}\,.$$

Note that $\#(A_1 \times \cdots \times A_n) = \#(A_1) \cdots \#(A_n)$.

Let S be another set, for example, $S = \mathbb{R}$, and let $f : M \to S$ be some mapping from M to S. Given a subset $B \subseteq S$, we denote the **preimage** of B with respect to f by

$$f^{-1}(B) := \{x \in M : f(x) \in B\}\,. \tag{A.1}$$

1 Sometimes called "mutually exclusive."

2 More precisely, one should say "pairwise disjoint."

In other words, an element $x \in M$ belongs to $f^{-1}(B)$ if and only if its image with respect to f is an element of B.

We summarize some crucial properties of the preimage in a proposition.

Proposition A.2.1. *Let $f : M \to S$ be a mapping from M into another set S.*
(1) $f^{-1}(\emptyset) = \emptyset$ *and* $f^{-1}(S) = M$.
(2) *For any subsets $B_j \subseteq S$ the following equalities are valid:*

$$f^{-1}\left(\bigcup_{j\geq 1} B_j\right) = \bigcup_{j\geq 1} f^{-1}(B_j) \text{ and } f^{-1}\left(\bigcap_{j\geq 1} B_j\right) = \bigcap_{j\geq 1} f^{-1}(B_j)\,. \tag{A.2}$$

Proof: We only prove the left-hand equality in eq. (A.2). The right-hand one is proved by the same methods. Furthermore, assertion (1) follows immediately.

Take $x \in f^{-1}\left(\bigcup_{j\geq 1} B_j\right)$. This happens if and only if

$$f(x) \in \bigcup_{j\geq 1} B_j \tag{A.3}$$

is satisfied. But this is equivalent to the existence of a certain $j_0 \geq 1$ with $f(x) \in B_{j_0}$. By definition of the preimage the last statement may be reformulated as follows: there exists a $j_0 \geq 1$ such that $x \in f^{-1}(B_{j_0})$. But this implies

$$x \in \bigcup_{j\geq 1} f^{-1}(B_j)\,. \tag{A.4}$$

Consequently, an element $x \in M$ satisfies condition (A.3) if and only if property (A.4) holds. This proves the left-hand identity in formulas (A.2). ∎

A.3 Combinatorics

A.3.1 Binomial Coefficients

A one-to-one mapping π from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$ is called a **permutation** (of order n). Any permutation reorders the numbers from 1 to n as $\pi(1), \pi(2), \ldots, \pi(n)$ and, vice versa, each reordering of these numbers generates a permutation. One way to write a permutations is

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{pmatrix}$$

For example, if $n = 3$, then $\pi = \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix}$ is equivalent to the order $2, 3, 1$ or to $\pi(1) = 2, \pi(2) = 3$ and $\pi(3) = 1$.

Let S_n be the set of all permutations of order n. Then one may ask for $\#(S_n)$ or, equivalently, for the number of possible orderings of the numbers $\{1, \dots, n\}$.

To treat this problem we need the following definition.

Definition A.3.1. For $n \in \mathbb{N}$ we define ***n*-factorial** by setting

$$n! = 1 \cdot 2 \cdots (n-1) \cdot n$$

Furthermore, let $0! = 1$.

Now we may answer the question about the cardinality of S_n.

Proposition A.3.2. *We have*

$$\#(S_n) = n! \tag{A.5}$$

or, equivalently, there are $n!$ different ways to order n distinguishable objects.

Proof: The proof is done by induction over n. If $n = 1$ then $\#(S_1) = 1 = 1!$ and eq. (A.5) is valid.

Now suppose that eq. (A.5) is true for n. In order to prove eq. (A.5) for $n+1$ we split S_{n+1} as follows:

$$S_{n+1} = \bigcup_{k=1}^{n+1} A_k,$$

where

$$A_k = \{\pi \in S_{n+1} : \pi(n+1) = k\}, \quad k = 1, \dots, n+1.$$

Each $\pi \in A_k$ generates a one-to-one mapping $\tilde{\pi}$ from $\{1, \dots, n\}$ onto the set $\{1, \dots, k-1, k+1, \dots, n\}$ by letting $\tilde{\pi}(j) = \pi(j)$, $1 \le j \le n$. Vice versa, each such $\tilde{\pi}$ defines a permutation $\pi \in A_k$ by setting $\pi(j) = \tilde{\pi}(j)$, $j \le n$, and $\pi(n+1) = k$. Consequently, since eq. (A.5) holds for n we get $\#(A_k) = n!$. Furthermore, the A_ks are disjoint, and

$$\#(S_{n+1}) = \sum_{k=1}^{n+1} \#(A_k) = (n+1) \cdot n! = (n+1)!\,,$$

hence eq. (A.5) also holds for $n+1$. This completes the proof. ∎

Next we treat a tightly related problem. Say we have n different objects and we want to distribute them into two disjoint groups, one having k elements, the other $n-k$. Hereby it is of no interest in which order the elements are distributed, only the composition of the two sets matters.

Example A.3.3. There are 52 cards in a deck that are distributed to two players, so that each of them gets 26 cards. For this game it is only important which cards each player has, not in which order the cards were received. Here $n = 52$ and $k = n - k = 26$.

The main question is: how many ways can n elements be distributed, say the numbers from 1 to n, into one group of k elements and into another of $n - k$ elements? In the above example, that is how many ways can 52 cards be distributed into two groups of 26.

To answer this question we use the following auxiliary model. Let us take any permutation $\pi \in S_n$. We place the numbers $\pi(1), \ldots, \pi(k)$ into group 1 and the remaining $\pi(k+1), \ldots, \pi(n)$ into group 2. In this way we obtain all possible distributions but many of them appear several times. Say two permutations π_1 and π_2 are equivalent if (as sets)

$$\{\pi_1(1), \ldots, \pi_1(k)\} = \{\pi_2(1), \ldots, \pi_2(k)\}\,.$$

Of course, this also implies

$$\{\pi_1(k+1), \ldots, \pi_1(n)\} = \{\pi_2(k+1), \ldots, \pi_2(n)\}\,,$$

and two permutations generate the same partition if and only if they are equivalent. Equivalent permutations are achieved by taking one fixed permutation π, then permuting $\{\pi(1), \ldots, \pi(k)\}$ and also $\{\pi(k+1), \ldots, \pi(n)\}$. Consequently, there are exactly $k!(n-k)!$ permutations that are equivalent to a given one. Summing up, we get that there are $\frac{n!}{k!(n-k)!}$ different classes of equivalent permutations. Setting

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

we see the following.

! There are $\binom{n}{k}$ different ways to distribute n objects into one group of k and into another one of $n - k$ elements.

The numbers $\binom{n}{k}$ are called **binomial coefficients**, read "n chosen k." We let $\binom{n}{k} = 0$ in case of $k > n$ or $k < 0$.

Example A.3.4. A digital word of length n consists of n zeroes or ones. Since at every position we may have either 0 or 1, there are 2^n different words of length n. How many of these words possess exactly k ones or, equivalently, $n-k$ zeroes? To answer this put all positions where there is a "1" into a first group and those where there is a "0" into a second one. In this way the numbers from 1 to n are divided into two different groups of size k and $n-k$, respectively. But we already know how many such partitions exist, namely $\binom{n}{k}$. As a consequence we get

---

There are $\binom{n}{k}$ words of length n possessing exactly k ones and $n-k$ zeroes.

The next proposition summarizes some crucial properties of binomial coefficients.

Proposition A.3.5. *Let n be a natural number, $k = 0, \dots, n$ and let $r \geq 0$ be an integer. Then the following equations hold:*

$$\binom{n}{k} = \binom{n}{n-k} \tag{A.6}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \textit{and} \tag{A.7}$$

$$\binom{n+r}{n} = \sum_{j=0}^{n} \binom{n+r-j-1}{n-j} = \sum_{j=0}^{n} \binom{r+j-1}{j}. \tag{A.8}$$

Proof: Equations (A.6) and (A.7) follow immediately by the definition of the binomial coefficients. Note that eq. (A.7) also holds if $k = n$ because we agreed that $\binom{n-1}{n} = 0$.

An iteration of eq. (A.7) leads to

$$\binom{n}{k} = \sum_{j=0}^{k} \binom{n-j-1}{k-j}.$$

Replacing in the last equation n by $n+r$ as well as k by n we obtain the left-hand identity (A.8). The right-hand equation follows by inverting the summation, that is, one replaces j by $n-j$. ∎

Remark A.3.6. Equation (A.7) allows a graphical interpretation by **Pascal's triangle**. The coefficient $\binom{n}{k}$ in the nth row follows by summing the two values $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$ above $\binom{n}{k}$ in the $(n-1)$th row.

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & 1 & & & & 1 & & & & & \\
 & & & & 1 & & & 2 & & & 1 & & & & \\
 & & & 1 & & 3 & & & & 3 & & 1 & & & \\
 & & \cdot & & \cdot & & & \cdot & & & \cdot & & \cdot & & \\
 & 1 & \cdot & \cdot & \cdot & \binom{n-1}{k-1} & & & & \binom{n-1}{k} & \cdot & \cdot & \cdot & 1 & \\
1 & \binom{n}{1} & \cdot & \cdot & \cdot & \cdot & & \binom{n}{k} & & \cdot & \cdot & \cdot & \cdot & \binom{n}{n-1} & 1
\end{array}$$

Next we state and prove the important binomial theorem.

Proposition A.3.7 (Binomial theorem). *For real numbers a and b and any $n \in \mathbb{N}_0$,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}\,. \tag{A.9}$$

Proof: The binomial theorem is proved by induction over n. If $n = 0$, then eq. (A.9) holds trivially.

Suppose now that eq. (A.9) has been proven for $n-1$. Our aim is to verify that it is also true for n. Using that the expansion holds for $n-1$ we get

$$\begin{aligned}
(a+b)^n &= (a+b)^{n-1}(a+b) \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} a^{k+1} b^{n-1-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} a^k b^{n-k} \\
&= a^n + \sum_{k=0}^{n-2} \binom{n-1}{k} a^{k+1} b^{n-1-k} + b^n + \sum_{k=1}^{n-1} \binom{n-1}{k} a^k b^{n-k} \\
&= a^n + b^n + \sum_{k=1}^{n-1} \left[\binom{n-1}{k-1} + \binom{n-1}{k} \right] a^k b^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}\,,
\end{aligned}$$

where we used eq. (A.7) in the last step . ■

The following property of binomial coefficients plays an important role when introducing the hypergeometric distribution (compare Proposition 1.4.24). It is also used during the investigation of sums of independent binomial distributed random variables (compare Proposition 4.6.1).

Proposition A.3.8 (Vandermonde's identity). *If k, m, and n in $\mathbb{N}_0$, then*

$$\sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j} = \binom{n+m}{k}. \tag{A.10}$$

Proof: An application of the binomial theorem leads to

$$(1+x)^{n+m} = \sum_{k=0}^{n+m} \binom{n+m}{k} x^k\,, \quad x \in \mathbb{R}\,. \tag{A.11}$$

On the other hand, another use of Proposition A.3.7 implies[3]

$$
\begin{aligned}
(1+x)^{n+m} &= (1+x)^n(1+x)^m \\
&= \left[\sum_{j=0}^{n}\binom{n}{j}x^j\right]\left[\sum_{i=0}^{m}\binom{m}{i}x^i\right] = \sum_{j=0}^{n}\sum_{i=0}^{m}\binom{n}{j}\binom{m}{i}x^{i+j} \\
&= \sum_{k=0}^{n+m}\left[\sum_{i+j=k}\binom{n}{j}\binom{m}{i}\right]x^k = \sum_{k=0}^{n+m}\left[\sum_{j=0}^{k}\binom{n}{j}\binom{m}{k-j}\right]x^k .
\end{aligned}
\tag{A.12}
$$

The coefficients in an expansion of a polynomial are unique. Hence, in view of eqs. (A.11) and (A.12), we get for all $k \le m+n$ the identity

$$
\binom{n+m}{k} = \sum_{j=0}^{k}\binom{n}{j}\binom{m}{k-j} .
$$

Hereby note that both sides of eq. (A.10) become zero whenever $k > n+m$. This completes the proof. ∎

Our next objective is to generalize the binomial coefficients. In view of

$$
\binom{n}{k} = \frac{n\,(n-1)\ \cdots\ (n-k+1)}{k!}
$$

for $k \ge 1$ and $n \in \mathbb{N}$ the **generalized binomial coefficient** is introduced as

$$
\binom{-n}{k} := \frac{-n\,(-n-1)\ \cdots\ (-n-k+1)}{k!} . \tag{A.13}
$$

The next lemma shows the tight relation between generalized and "ordinary" binomial coefficients.

Lemma A.3.9. *For $k \ge 1$ and $n \in \mathbb{N}$,*

$$
\binom{-n}{k} = (-1)^k\binom{n+k-1}{k} .
$$

3 When passing from line 2 to line 3 the order of summation is changed. One no longer sums over the rectangle $[0,m]\times[0,n]$. Instead one sums along the diagonals, where $i+j=k$.

Proof: By definition of the generalized binomial coefficient we obtain

$$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!}$$
$$= (-1)^k \frac{(n+k-1)(n+k-2)\cdots(n+1)\,n}{k!} = (-1)^k \binom{n+k-1}{k}.$$

This completes the proof. ∎

For example, Lemma A.3.9 implies $\binom{-1}{k} = (-1)^k$ and $\binom{-n}{1} = -n$.

A.3.2 Drawing Balls out of an Urn

Assume that there are n balls labeled from 1 to n in an urn. We draw k balls out of the urn, thus observing a sequence of length k with entries from $\{1, \ldots, n\}$. How many different results (sequences) may be observed? To answer this question we have to decide the arrangement of drawing. Do we or do we not replace the chosen ball? Is it important in which order the balls were chosen or is it only of importance which balls were chosen at all? Thus, we see that there are four different ways to answer this question (replacement or nonreplacement, recording the order or nonrecording).

Example A.3.10. Let us regard the drawing of two balls out of four, that is, $n = 4$ and $k = 2$. Depending on the different arrangements the following results may be observed. Note, for example, that in the two latter cases (3, 2) does not appear because it is identical to (2, 3).

Replacement and order is important

(1, 1)	(1, 2)	(1, 3)	(1, 4)
(2, 1)	(2, 2)	(2, 3)	(2, 4)
(3, 1)	(3, 2)	(3, 3)	(3, 4)
(4, 1)	(4, 2)	(4, 3)	(4, 4)

16 different results

Nonreplacement and order is important

·	(1, 2)	(1, 3)	(1, 4)
(2, 1)	·	(2, 3)	(2, 4)
(3, 1)	(3, 2)	·	(3, 4)
(4, 1)	(4, 2)	(4, 3)	·

12 different results

Replacement and order is not important

(1, 1)	(1, 2)	(1, 3)	(1, 4)
·	(2, 2)	(2, 3)	(2, 4)
·	·	(3, 3)	(3, 4)
·	·	·	(4, 4)

10 different results

Nonreplacement and order is not important

·	(1, 2)	(1, 3)	(1, 4)
·	·	(2, 3)	(2, 4)
·	·	·	(3, 4)
·	·	·	·

6 different results

Let us come back now to the general situation of n different balls from which we choose k at random.

Case 1 : **Drawing with replacement and taking the order into account.**

We have n different possibilities for the choice of the first ball and since the chosen ball is placed back there are also n possibilities for the second one and so on. Thus, there are n possibilities for each of the k balls, leading to the following result.

!

The number of different results in this case is n^k

Example A.3.11. Letters in Braille, a scripture for blind people, are generated by dots or nondots at six different positions. How many letters may be generated in that way?

Answer: It holds that $n = 2$ (dot or no dot) at $k = 6$ different positions. Hence, the number of possible representable letters is $2^6 = 64$. In fact, there are only 63 possibilities because we have to rule out the case of no dots at all 6 positions.

Case 2 : **Drawing without replacement and taking the order into account.**

This case only makes sense if $k \le n$. There are n possibilities to choose the first ball. After that there are still $n - 1$ balls in the urn. Hence there are only $n - 1$ possibilities for the second choice, $n - 2$ for the third, and so on. Summing up we get the following.

!

The number of possible results in this case equals

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Example A.3.12. In a lottery 6 numbers are chosen out of 49. Of course, the chosen numbers are not replaced. If we record the numbers as they appear (not putting them in order) how many different sequences of six numbers exist?

Answer: Here we have $n = 49$ and $k = 6$. Hence the wanted number equals

$$\frac{49!}{43!} = 49 \cdots 44 = 10{,}068{,}347{,}520$$

Case 3 : **Drawing with replacement not taking the order into account.**

This case is more complicated and requires a different point of view. We count how often each of the n balls was chosen during the k trials. Let $k_1 \ge 0$ be the frequency of the first ball, $k_2 \ge 0$ that of the second one, and so on. In this way we obtain n non-negative integers $k_1, \ldots, k_n$ satisfying

$$k_1 + \cdots + k_n = k\,.$$

Indeed, since we choose k balls, the frequencies have to sum to k. Consequently, the number of possible results when drawing k of n balls with replacement and not taking the order into account coincides with

$$\#\{(k_1,\dots,k_n)\,,\, k_j \in \mathbb{N}_0\,,\, k_1+\cdots+k_n = k\}\,. \tag{A.14}$$

In order to determine the cardinality (A.14) we use the following auxiliary model:

Let $B_1,\dots,B_n$ be n boxes. Given n nonnegative integers $k_1,\dots,k_n$, summing to k, we place exactly k_1 dots into B_1, k_2 dots into B_2, and so on. At the end we distributed k nondistinguishable dots into n different boxes. Thus, we see that the value of (A.14) coincides with the number of different possibilities to distribute k nondistinguishable dots into n boxes. Now assume that the boxes are glued together; on the very left we put box B_1, on its right we put box B_2 and continue in this way up to box B_n on the very right. In this way we obtain $n+1$ dividing walls, two outer and $n-1$ inner ones. Now we get all possible distributions of k dots into n boxes by shuffling the k dots and the $n-1$ inner dividing walls. For example, if we get the order $w, w, d, d, w\dots$, then this means that there are no dots in B_1 and B_2, but there are two dots in B_3.

Summing up, we have $N = n+k-1$ objects, k of them are dots and $n-1$ are walls. As we know there are $\binom{N}{k}$ different ways to order these N objects. Hence we arrived at the following result.

The number of possibilities to distribute k anonymous dots into n boxes equals !

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}\,.$$

It coincides with $\#\{(k_1,\dots,k_n)\,,\, k_j \in \mathbb{N}_0\,,\, k_1+\cdots+k_n = k\}$ as well as with the number of different results when choosing k balls out of n **with replacement and not taking order into account.**

Example A.3.13. Dominoes are marked on each half either with no dots, one dot or up to six dots. Hereby the dominoes are symmetric, that is, a tile with three dots on the left-hand side and two ones on the right-hand one is identical with one having two dots on the left-hand side and three dots on the right-hand one. How many different dominoes exist?

Answer: It holds $n = 7$ and $k = 2$, hence the number of different dominoes equals

$$\binom{7+2-1}{2} = \binom{8}{2} = 28\,.$$

Case 4 : **Drawing without replacement not taking the order into account.**

Here we also have to assume $k \leq n$. We already investigated this case when we introduced the binomial coefficients. The k chosen numbers are put in group 1, the remaining $n - k$ balls in group 2. As we know there are $\binom{n}{k}$ ways to split the n numbers into such two groups. Hence we obtained the following.

! The number of different results in this case is $\binom{n}{k}$

Example A.3.14. If the order of the six numbers is not taken into account in Example A.3.12, that is, we ignore which number was chosen first, which second, and so on the number of possible results equals

$$\binom{49}{6} = \frac{49 \cdots 43}{6!} = 13,983,816$$

Let us summarize the four different cases in a table. Here **O** and **NO** stand for recording or nonrecording of the order while **R** and **NR** represent replacement or nonreplacement.

	R	**NR**
O	n^k	$\frac{n!}{(n-k)!}$
NO	$\binom{n+k-1}{k}$	$\binom{n}{k}$

A.3.3 Multinomial Coefficients

The binomial coefficient $\binom{n}{k}$ describes the number of possibilities to distribute n objects into two groups of k and $n - k$ elements. What happens if we have not only two groups but $m \geq 2$? Say the first group has k_1 elements, the second has k_2 elements, and so on, up to the mth group that has k_m elements. Of course, if we distribute n elements the k_j have to satisfy

$$k_1 + \cdots + k_m = n\,.$$

Using exactly the same arguments as in the case where $m = 2$ we get the following.

! There exists exactly $\frac{n!}{k_1! \cdots k_m!}$ different ways to distribute n elements into m groups of sizes $k_1, k_2, \ldots, k_m$ where $k_1 + \cdots + k_m = n$.

In accordance with the binomial coefficient we write

$$\binom{n}{k_1, \ldots, k_m} := \frac{n!}{k_1! \cdots k_m!}\,, \quad k_1 + \cdots + k_m = n\,, \tag{A.15}$$

and call $\binom{n}{k_1,\ldots,k_m}$ a **multinomial coefficient**, read "n chosen k_1 up to k_m."

Remark A.3.15. If $m = 2$, then $k_1 + k_2 = n$, and

$$\binom{n}{k_1, k_2} = \binom{n}{k_1, n - k_1} = \binom{n}{k_1} = \binom{n}{k_2}.$$

Example A.3.16. A deck of cards for playing skat consists of 32 cards. Three players each gets 10 cards; the remaining two cards (called "skat") are placed on the table. How many different distributions of the cards exist?

Answer: Let us first define what it means for two distribution of cards to be identical. Say, this happens if each of the three players has exactly the same cards as in the previous game. Therefore, the remaining two cards on the table are also identical. Hence we distribute 32 cards into 4 groups possessing $10, 10, 10$, and 2 elements. Consequently, the number of different distributions equals [4]

$$\binom{32}{10, 10, 10, 2} = \frac{32!}{(10!)^3\, 2!} = 2.753294409 \times 10^{15}\,.$$

Remark A.3.17. One may also look at multinomial coefficients from a different point of view. Suppose we are given n balls of m different colors. Say there are k_1 balls of color 1, k_2 balls of color 2, up to k_m balls of color m where, of course, $k_1 + \cdots + k_m = n$. Then there exist

$$\binom{n}{k_1, \ldots, k_m}$$

different ways to order these n balls. This is followed by the same arguments as we used in Example A.3.4 for $m = 2$.

For instance, given 3 blue, 4 red and 2 white balls, then there are

$$\binom{9}{3, 4, 2} = \frac{9!}{3!\, 4!\, 2!} = 1260$$

different ways to order them.

Finally, let us still mention that in the literature one sometimes finds another (equivalent) way for the introduction of the multinomial coefficients. Given nonnegative integers $k_1, \ldots, k_m$ with $k_1 + \cdots + k_m = n$, it follows that

$$\binom{n}{k_1, \ldots, k_m} = \binom{n}{k_1}\binom{n - k_1}{k_2}\binom{n - k_1 - k_2}{k_3} \cdots \binom{n - k_1 - \cdots - k_{m-1}}{k_m}. \tag{A.16}$$

A direct proof of this fact is easy and left as an exercise.

4 The huge size of this number explains why playing skat never becomes boring.

There is a combinatorial interpretation of the expression on the right-hand side of eq. (A.16). To reorder n balls of m different colors, one chooses first the k_1 positions for balls of color 1. There are $\binom{n}{k_1}$ ways to do this. Thus, there remain $n - k_1$ possible positions for balls of color 2, and there are $\binom{n-k_1}{k_2}$ possible choices for this, and so on. Note that at the end there remain k_m positions for k_m balls; hence, the last term on the right-hand side of eq. (A.16) equals 1.

Let us come now to the announced generalization of Proposition A.3.7.

Proposition A.3.18 (Multinomial theorem). *Let $n \geq 0$. Then for any $m \geq 1$ and real numbers $x_1, \ldots, x_m$,*

$$(x_1 + \cdots + x_m)^n = \sum_{\substack{k_1+\cdots+k_m=n \\ k_i \geq 0}} \binom{n}{k_1, \ldots, k_m} x_1^{k_1} \cdots x_m^{k_m} . \tag{A.17}$$

Proof: Equality (A.17) is proved by induction. In contrast to the proof of the binomial theorem, now induction is done over m, the number of summands.

If $m = 1$ the assertion is valid by trivial reasons.

Suppose now eq. (A.17) holds for m, all $n \geq 1$ and all real numbers $x_1, \ldots, x_m$. We have to show the validity of eq. (A.17) for $m + 1$ and all $n \geq 1$. Given real numbers $x_1, \ldots, x_{m+1}$ and $n \geq 1$ set $y := x_1 + \cdots + x_m$. Using A.3.7, by the validity of eq. (A.17) for m and all $n - j$, $0 \leq j \leq n$, we obtain

$$(x_1 + \cdots + x_{m+1})^n = (y + x_{m+1})^n = \sum_{j=1}^{n} \frac{n!}{j!\,(n-j)!}\, x_{m+1}^{j} y^{n-j}$$

$$= \sum_{j=1}^{n} \frac{n!}{j!\,(n-j)!} \sum_{\substack{k_1+\cdots+k_m=n-j \\ k_i \geq 0}} \frac{(n-j)!}{k_1! \cdots k_m!}\, x_1^{k_1} \cdots x_m^{k_m} x_{m+1}^{j} .$$

Replacing j by k_{m+1} and combining both sums leads to

$$(x_1 + \cdots + x_{m+1})^n = \sum_{\substack{k_1+\cdots+k_{m+1}=n \\ k_i \geq 0}} \frac{n!}{k_1! \cdots k_{m+1}!}\, x_1^{k_1} \cdots x_{m+1}^{k_{m+1}} ,$$

hence eq. (A.17) is also valid for $m + 1$. This completes the proof. ∎

Remark A.3.19. The number of summands in eq. (A.17) equals[5] $\binom{n+m-1}{n}$.

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5 Compare case 3 in Section A.3.2.