Mathematical Statistics, Winter semester 2020/21
Problem sheet 5

14) Let $X_1, \ldots, X_n$ be i.i.d. with $P_\theta(X_i = 1) = \theta = 1 - P_\theta(X_i = 0)$ and let $\widehat{\theta}_{\alpha,\beta} = \frac{\sum_{i=1}^n X_i + \alpha}{n + \alpha + \beta}$ be the Bayesian estimator mean squared error risk and prior distribution $\pi = \mathrm{Beta}(\alpha, \beta)$.)

   (i) Show that $\widehat{\theta}_{\alpha,\beta}$ is admissible for all $\alpha, \beta > 0$.

   (ii) Show that $\widehat{\theta}_{\sqrt{n}/2,\sqrt{n}/2}$ has a constant risk function. Is $\widehat{\theta}_{\sqrt{n}/2,\sqrt{n}/2}$ a minimax estimator in $\Theta = [0, 1]$?

15) Let $X_1, \ldots, X_n$ be i.i.d. with $X_i \sim \mathrm{Bin}(1, \theta)$, where $\theta \in \Theta := \{\theta_0, \theta_1\} \subseteq (0, 1)$, $\theta_0 \neq \theta_1$.

For $\beta \in [0, 1]$, find a (possibly randomized) test φ which minimizes

$$\beta \, E_{\theta_0}[\varphi(X)] \; + \; (1 - \beta) \, E_{\theta_1}[1 - \varphi(X)].$$

$(X = (X_1, \ldots, X_n)^T)$

16) Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim \mathcal{N}(\theta, 1)$, $i = 1, \ldots, n$. Consider the problem of testing the following hypotheses.

$$H_0\colon \quad \theta = \theta_0 \qquad \text{vs.} \qquad H_1\colon \quad \theta = \theta_1,$$

where $\theta_0 < \theta_1$.

How large must the sample size n be in order that the probabilities of type I and type II errors are both not greater than 0.05?

Hint: It holds that $\Phi^{-1}(0.95) \approx 1.64$.