Measure Theory, Winter semester 2021/22
Solutions to Problem sheet 1

1) Let Ω be a non-empty set and let $\mathcal{A}$ be a σ-algebra on Ω. Suppose that P_1 and P_2 are probability measures on $(\Omega, \mathcal{A})$, i.e. $P_i\colon \mathcal{A} \to [0,1]$ satisfies

(i) $P_i(\emptyset) = 0$, $P_i(\Omega) = 1$,

(ii) if $A_1, A_2, \ldots$ are disjoint sets that belong to $\mathcal{A}$, then

$$P_i\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P_i(A_i).$$

Show that the collection of sets $\mathcal{D} := \{A \in \mathcal{A}\colon P_1(A) = P_2(A)\}$ is a Dynkin system on Ω.
Hint: Note that $P_i(A^c) = P_i(\Omega \setminus A) = P_i(\Omega) - P_i(A)$.

Solution

We verify that the system $\mathcal{D}$ satisfies the axioms of a Dynkin system on Ω:

a) Since $P_1(\Omega) = 1 = P_2(\Omega)$ we have that

$$\Omega \in \mathcal{D}.$$

b) Suppose that $A \in \mathcal{D}$. Then $P_1(A) = P_2(A)$, which implies by (i) that

$$P_1(A^c) = P_1(\Omega) - P_1(A) = P_2(\Omega) - P_2(A) = P_2(A^c).$$

Hence,

$$A \in \mathcal{D} \qquad \text{implies that} \qquad A^c \in \mathcal{D}.$$

c) Suppose that $A_1, A_2, \ldots$ are disjoint sets that belong to $\mathcal{D}$, i.e. $P_1(A_i) = P_2(A_i)$ for all $i \in \mathbb{N}$. Then

$$P_1\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P_1(A_i) = \sum_{i=1}^{\infty} P_2(A_i) = P_2\Big(\bigcup_{i=1}^{\infty} A_i\Big).$$

Hence,

$$A_1, A_2, \ldots \in \mathcal{D} \quad \text{for disjoint sets implies that} \qquad \bigcup_{i=1}^{\infty} A_i \in \mathcal{D}.$$

It follows that $\mathcal{D}$ is a Dynkin system on Ω.

2) Let Ω be a non-empty set and let $(\mathcal{A}_i)_{i\in I}$ be a non-empty collection of σ-algebras on Ω, where I is an arbitrary (finite, countably infinite or even uncountable) index set.

Show that the intersection of these σ-algebras,

$$\bigcap_{i\in I} \mathcal{A}_i = \{A \subseteq \Omega\colon \quad A \in \mathcal{A}_i \quad \text{for all } i \in I\},$$

is a σ-algebra on Ω.

Solution

Let $\mathcal{A} = \bigcap_{i\in I} \mathcal{A}_i$ We verify that $\mathcal{A}$ satisfies the axions of a σ-algebra on Ω:

a) Since $\Omega \in \mathcal{A}_i$ for all $i \in I$ we have that

$$\Omega \in \mathcal{A}.$$

b) Let $A \in \mathcal{A}$ be arbitrary. Then $A \in \mathcal{A}_i$ for all $i \in I$, and so $A^c \in \mathcal{A}_i$ for all $i \in I$. Hence,

$$A \in \mathcal{D} \qquad \text{implies that} \qquad A^c \in \mathcal{D}.$$

c) Suppose that $A_1, A_2, \ldots$ are arbitrary sets that belong to $\mathcal{A}$. Then, for all $i \in I$, $A_1, A_2, \ldots \in \mathcal{A}_i$, which implies $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_i$ for all $i \in I$, i.e. $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. Hence,

$$A_1, A_2, \ldots \in \mathcal{A} \quad \text{implies that} \qquad \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}.$$

It follows that $\mathcal{A}$ is a σ-algebra on Ω.