Measure Theory, Winter semester 2021/22
Solutions to Problem sheet 2

3) Let $\Omega = \mathbb{N}$ and let $\mathcal{C} := \{A \subseteq \Omega\colon\ A \text{ or } A^c \text{ is a finite subset of } \Omega\}$.

   (i) Show that $\mathcal{C}$ is a ring but not a σ-algebra on Ω.

   (ii) For $A \in \mathcal{C}$, let $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if A is infinite.
   Show that μ is a content on $(\Omega, \mathcal{C})$. Is μ a pre-measure on $(\Omega, \mathcal{C})$?

Solution

(i) We verify that the axioms of a ring are fulfilled by $\mathcal{C}$:

a) $\emptyset \in \mathcal{C}$ since $\emptyset$ is a finite set.

b) Let $A, B \in \mathcal{C}$. We have to show that $A \setminus B = A \cap B^c$ belongs to $\mathcal{C}$.
Case 1: If at least one of the sets A and B^c is finite, then $A \cap B^c$ is finite, and so $A \setminus B \in \mathcal{C}$.
Case 2: If both sets A and B^c are infinite, then A^c and B are finite. In this case $(A \setminus B)^c = (A \cap B^c)^c = A^c \cup B$ is finite; hence $A \setminus B \in \mathcal{C}$.

c) Let $A, B \in \mathcal{C}$.
Case 1: If both A and B are finite, then $A \cup B$ is also finite. Hence, $A \cup B \in \mathcal{C}$.
Case 2: If at least one of the sets A and B is infinite, then at least one of A^c and B^c is finite. Hence, $(A \cup B)^c$ is finite, and so $A \cup B \in \mathcal{C}$.

$\mathcal{C}$ is not a σ-algebra on $\mathbb{N}$ since $\{1\}, \{3\}, \ldots \in \mathcal{C}$ but $\bigcup_{i=1}^{\infty}\{2i-1\} \notin \mathcal{C}$.

(ii) It is easy to see that μ is a content on $\mathcal{C}$:
μ is a non-negative function and $\mu(\emptyset) = 0$.
Let A_1 and A_2 be disjoint sets that belong to $\mathcal{C}$.
Case 1: If both A and B are finite, then $\mu(A \cup B) = 0 = \mu(A) + \mu(B)$.
Case 2: If one of the sets A and B is infinite and the other set is finite, then $\mu(A \cup B) = 1 = \mu(A) + \mu(B)$.
Both sets A and B cannot be infinite simultaneously. Since $A \cap B = \emptyset$ we have that $A \subseteq B^c$. If A is infinite, then B^c is also infinite which implies that B is finite.

μ is not a pre-measure on $\mathcal{C}$. To disprove σ-additivity, consider the singletons $A_i = \{i\}$. Then $\bigcup_{i=1}^{\infty}\{i\} = \mathbb{N} \in \mathcal{C}$, but $\mu(\bigcup_{i=1}^{\infty}\{i\}) = 1 > 0 = \sum_{i=1}^{n} \mu(\{i\})$.

4) Let $P\colon \mathcal{B}^d \to [0,1]$ be a probability measure on $(\mathbb{R}^d, \mathcal{B}^d)$, and let $F\colon \mathbb{R}^d \to [0,1]$ be the corresponding distribution function, i.e.

$$F(x_1,\dots,x_d) \;=\; P((-\infty,x_1]\times\cdots\times(-\infty,x_d]) \qquad \forall x_1,\dots,x_d\in\mathbb{R}.$$

Show that, for all $x_1,\dots,x_d\in\mathbb{R}$, $y_1,\dots,y_d\geq 0$,

$$P((x_1,x_1+y_1]\times\cdots\times(x_d,x_d+y_d]) \;=\; \sum_{(\theta_1,\dots,\theta_d)\in\{0,1\}^d} (-1)^{\sum_{i=1}^d(1-\theta_i)} F(x_1+\theta_1 y_1,\dots,x_d+\theta_d y_d).$$

Hint: Consider the sets $A_i \;=\; (-\infty,x_1+y_1]\times\cdots\times(-\infty,x_{i-1}+y_{i-1}]\times(-\infty,x_i]\times(-\infty,x_{i+1}+y_{i+1}]\times\cdots\times(-\infty,x_d+y_d]$.

Solution

Let $A_i = (-\infty,x_1+y_1]\times\cdots\times(-\infty,x_{i-1}+y_{i-1}]\times(-\infty,x_i]\times(-\infty,x_{i+1}+y_{i+1}]\times\cdots\times(-\infty,x_d+y_d]$.

We obtain by the inclusion-exclusion principle that

$$\begin{aligned}
P(A_1\cup\cdots\cup A_d) \;&=\; \sum_{k=1}^d(-1)^{k+1}\sum_{1\leq i_1<\ldots<i_k\leq d} P(A_{i_1}\cap\cdots\cap A_{i_k}) \\
&=\; \sum_{\theta\in\{0,1\}^d,\,\theta\neq(1,\dots,1)} (-1)^{\sum_{i=1}^d(1-\theta_i)\,+\,1} F(x_1+\theta_1 y_1,\dots,x_d+\theta_d y_d).
\end{aligned}$$

Then

$$\begin{aligned}
&P((x_1,x_1+y_1]\times\cdots\times(x_d,x_d+y_d]) \\
&\quad=\; P((-\infty,x_1+y_1]\times\cdots\times(-\infty,x_d+y_d]) \;-\; P(A_1\cup\cdots\cup A_d) \\
&\quad=\; (-1)^{d+d}F(x_1+y_1,\dots,x_d+y_d) \\
&\qquad\; + \sum_{\theta\in\{0,1\}^d,\,\theta\neq(1,\dots,1)} (-1)^{\sum_{i=1}^d(1-\theta_i)} F(x_1+\theta_1 y_1,\dots,x_d+\theta_d y_d).
\end{aligned}$$