Measure Theory, Winter semester 2021/22
Solutions to Problem sheet 3

5) Let $\lambda^*\colon 2^{\mathbb{R}} \to [0,\infty]$ be the outer measure on $\mathbb{R}$ defined by

$$\lambda^*(Q) \;=\; \inf\Big\{\sum_{i=1}^{\infty} \lambda_0^1(A_i)\colon \qquad A_1, A_2, \ldots \in \mathcal{B}_0^1,\; Q \subseteq \bigcup_{i=1}^{\infty} A_i\Big\}.$$

Using **only the definition** of λ^*, show that $\lambda^*(C) = 0$ if C is a countable subset of $\mathbb{R}$.

Solution
Let C be a countable subset of $\mathbb{R}$, i.e. $C = \{r_1, r_2, \ldots\}$. Let $\epsilon > 0$ be arbitrary. Define for each $i \in \mathbb{N}$

$$A_i \;=\; (r_i - \epsilon 2^{-i}, r_i].$$

Then

$$C \;\subseteq\; \bigcup_{i=1}^{\infty} A_i, \qquad A_i \in \mathcal{I}_1 \subseteq \mathcal{B}_0^1,$$

and

$$\sum_{i=1}^{\infty} \lambda_0^1(A_i) \;=\; \sum_{i=1}^{\infty} \epsilon\, 2^{-i} \;=\; \epsilon.$$

Since $\lambda^*(C) \le \sum_{i=1}^{\infty} \lambda_0^1(A_i)$ we obtain that $\lambda^*(C) = 0$.

6) *(A model for countably many fair coin tosses)*
Let

$$\begin{aligned}\Omega &= \{\omega = (\omega_1, \omega_2, \ldots)\colon \quad \omega_i \in \{0,1\} \text{ for all } i \in \mathbb{N}\},\\ \mathcal{C}_n &= \{A \times \Omega\colon \quad A \subseteq \{0,1\}^n\} = \Big\{\{\omega \in \Omega\colon \quad (\omega_1, \ldots .\omega_n) \in A\}\Big|\ A \subseteq \{0,1\}^n\Big\}\end{aligned}$$

and

$$\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n.$$

Define $P_0\colon \mathcal{C} \to [0,1]$ such that, for $A \in \{0,1\}^n$ and $n \in \mathbb{N}$,

$$P_0(A \times \Omega) = P_0(\{\omega \in \Omega\colon (\omega_1, \ldots, \omega_n) \in A\}) = \frac{\#A}{2^n}.$$

(i) Show that $\mathcal{C}$ is an algebra (and therefore a ring) on Ω.

(ii) For each $c \in [0,1]$, let $B_c = \{\omega \in \Omega\colon \frac{1}{n}\sum_{i=1}^n \omega_i \underset{n\to\infty}{\longrightarrow} c\}$.
Show that $B_c \in \sigma(\mathcal{C})$.
(*Hint: Use that* $\{\omega \in \Omega\colon |\frac{1}{n}\sum_{i=1}^n \omega_i - c| \leq \frac{1}{m}\} \in \mathcal{C}$.)

(iii) Let $A \times \Omega = B \times \Omega$, where $A \subseteq \{0,1\}^m$ and $B \subseteq \{0,1\}^n$.
Show that

$$P_0(A \times \Omega) = P_0(B \times \Omega).$$

(iv) Show that P_0 is a content on $\mathcal{C}$.
(*Actually, it can also be shown that* P_0 *is a pre-measure on* $\mathcal{C}$.)

Solution

(i) We verify that $\mathcal{C}$ satisfies the axioms of an algebra on Ω:

a) Since $\Omega = \{0,1\} \times \Omega$ we have $\Omega \in \mathcal{C}_n\ \forall n \in \mathbb{N}$, and so $\Omega \in \mathcal{C}$.

b) Let $B \in \mathcal{C}$. Then B has the representation $B = A \times \Omega$, where $A \subseteq \{0,1\}^n$ for some $n \in \mathbb{N}$. Since $B^c = A^c \times \Omega$ we obtain that $B^c \in \mathcal{C}$.

c) Let $B_1, B_2 \in \mathcal{C}$. Then

$$B_1 = A_1 \times \Omega \qquad \text{and} \qquad B_2 = A_2 \times \Omega,$$

where $A_1 \subseteq \{0,1\}^m$ and $A_2 \subseteq \{0,1\}^n$. Let without loss of generality $m \geq n$. Then $B_2 = \underbrace{A_2 \times \{0,1\}^{m-n}}_{=:A_3} \times\Omega$, and so

$$B_1 \cup B_2 = (A_1 \cup A_3) \times \Omega,$$

i.e. $B_1 \cup B_2 \in \mathcal{C}$.

(ii) This follows from

$$B_c = \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \underbrace{\left\{ \omega \in \Omega\colon \quad \left| \frac{1}{n} \sum_{i=1}^{n} \omega_i \, - \, c \right| \, \leq \, \frac{1}{m} \right\}}_{\in \mathcal{C}_n = \mathcal{C}}.$$

(iii) Suppose that $A \times \Omega = B \times \Omega$, where $A \subseteq \{0,1\}^m$ and $B \subseteq \{0,1\}^n$. Let w.l.o.g. $m \geq n$. Then it follows from $A \times \Omega = B \times \Omega$ that

$$A \, = \, B \times \{0,1\}^{m-n}.$$

This implies

$$\begin{aligned} P_0(A \times \Omega) \quad &= \quad P_0(\underbrace{B \times \{0,1\}^{m-n}}_{\subseteq \{0,1\}^m} \times \Omega) \, = \, \frac{\#(B \times \{0,1\}^{m-n}}{2^m} \\ &= \quad \frac{\#B}{2^n} \, = \, P_0(B \times \Omega). \end{aligned}$$

(iv) We verify that P_0 satisfies the axioms of a content on $\mathcal{C}$:

a) P_0 is obviously a non-negative set function on $\mathcal{C}$.

b) Let $\emptyset_\Omega$ and $\emptyset_{\{0,1\}}$ denote the respective empty sets in Ω and $\{0,1\}$. Then

$$P_0(\emptyset_\Omega) \, = \, P_0(\emptyset_{\{0,1\}} \times \Omega) \, = \, 0.$$

c) Let A_1 and A_2 be disjoint sets that belong to $\mathcal{C}$. Then $A_1 = B_1 \times \Omega$ and $A_2 = B_2 \times \Omega$, where $B_1 \subseteq \{0,1\}^m$ and $B_2 \subseteq \{0,1\}^n$. Let w.l.o.g. $m \geq n$. Then $A_2 = \underbrace{(B_2 \times \{0,1\}^{m-n})}_{=: B_3} \times \Omega$. Since $A_1 \cap A_2 = \emptyset$ implies that B_1 and B_3 are disjoint we obtain that

$$P_0(A_1 \cup A_2) \, = \, \frac{\#(B_1 \cup B_3)}{2^m} \, = \, \frac{\#B_1}{2^m} \, + \, \underbrace{\frac{\#B_3}{2^m}}_{= \#B_2/2^n} \, = \, P_0(A_1) \, + \, P_0(A_2).$$