Measure Theory, Winter semester 2021/22
Solutions to Problem sheet 4

7) Consider the measure space $(\mathbb{R}, \mathcal{B}, \delta_0)$, where δ_0 denotes the Dirac measure concentrated at 0, i.e.

$$\delta_0(A) = \begin{cases} 1 & \text{if } 0 \in A, \\ 0 & \text{if } 0 \notin A. \end{cases}$$

Find the completion of $\mathcal{B}$.

Solution

First we identify the collection $\mathcal{N}$ of δ_0-null sets:
For $A \in \mathcal{B}$, we have that $\delta_0(A) = 0$ if and only if $0 \notin A$. Therefore,

$$\mathcal{N} = \{N \subseteq \mathbb{R}: \quad 0 \notin N\}.$$

The completion $\widetilde{\mathcal{B}}$ of $\mathcal{B}$ is given by

$$\widetilde{\mathcal{B}} = \{A \cup N: \quad A \in \mathcal{B},\, N \in \mathcal{N}\}.$$

Since the singleton $\{0\}$ belongs to $\mathcal{B}$ we obtain that

$$\widetilde{\mathcal{B}} = 2^{\mathbb{R}} = \{A: \quad A \subseteq \mathbb{R}\}.$$

8) Show that there is a Lebesgue measurable subset of $\mathbb{R}^2$ whose projection on $\mathbb{R}$ under the map $(x, y) \mapsto x$ is not Lebesgue measurable.

Solution

Let E be the set described in the proof of Corollary 1.4.2. Then $E \subseteq \mathbb{R}$ but $E \notin \mathcal{M}_{\lambda^*}$. Now we consider the set

$$F := E \times \{0\}.$$

Then $F \subseteq \mathbb{R} \times \{0\}$, $\lambda^2(\mathbb{R} \times \{0\}) = 0$. Since $\mathcal{M}_{\lambda^*}$ is complete we see that F is a Lebesgue measurable subset of $\mathbb{R}^2$.

Let $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate, i.e. $\pi((x, y)) = x$. Then the projection of the set F under the map π is given by

$$\pi(F) = \{\pi((x, y))\colon \quad (x, y) \in F\} = E.$$