Measure Theory, Winter semester 2021/22
Solutions to Problem sheet 5

9) Let $(\Omega, \mathcal{A})$ be a measurable space, and let $f, g\colon \Omega \to \mathbb{R}$ be $(\mathcal{A}-\mathcal{B})$-measurable functions. Show that the sets $\{\omega \in \Omega\colon f(\omega) < g(\omega)\}$ and $\{\omega \in \Omega\colon f(\omega) = g(\omega)\}$ belong to $\mathcal{A}$.

Solution
We have that

$$\begin{aligned} \{\omega \in \Omega\colon f(\omega) < g(\omega)\} &= \bigcup_{r \in \mathbb{Q}} \{\omega\colon f(\omega) < r,\ r < g(\omega)\} \\ &= \bigcup_{r \in \mathbb{Q}} \underbrace{\{\omega\colon f(\omega) < r\}}_{\in \mathcal{A}} \cap \underbrace{\{\omega\colon r < g(\omega)\}}_{\in \mathcal{A}}, \end{aligned}$$

which implies that

$$\{\omega \in \Omega\colon f(\omega) < g(\omega)\} \in \mathcal{A}.$$

We can show in literally the same way that

$$\{\omega \in \Omega\colon f(\omega) > g(\omega)\} \in \mathcal{A},$$

which also implies

$$\begin{aligned} \{\omega \in \Omega\colon f(\omega) = g(\omega)\} &= \{\omega \in \Omega\colon f(\omega) \neq g(\omega)\}^c \\ &= \Big(\{\omega \in \Omega\colon f(\omega) < g(\omega)\} \cup \{\omega \in \Omega\colon f(\omega) > g(\omega)\}\Big)^c \in \mathcal{A}. \end{aligned}$$

10) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable everywhere on $\mathbb{R}$.

Show that f' is $(\mathcal{B} - \mathcal{B})$-measurable.

Solution

Let, for $n \in \mathbb{N}$,

$$f_n(x) \;:=\; \frac{f(x + 1/n) \;-\; f(x)}{1/n}.$$

Since $x \mapsto x + 1/n$ is $(\mathcal{B} - \mathcal{B})$-measurable, the composition $x \mapsto f(x + 1/n)$ is also $(\mathcal{B} - \mathcal{B})$-measurable. Hence, f_n is the difference of two $(\mathcal{B} - \mathcal{B})$-measurable functions, multiplied by n, and therefore also $(\mathcal{B} - \mathcal{B})$-measurable.

Since $f'(x) = \lim_{n\to\infty} f_n(x)$ it follows that f' is $(\mathcal{B} - \mathcal{B})$-measurable.

11) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $f\colon \Omega \to \{0, 1, 2, \ldots\}$ be a non-negative integer-valued $(\mathcal{A} - \mathcal{B})$-measurable function.

Show that $\int_\Omega f \, d\mu \; = \; \sum_{n=1}^{\infty} \mu(\{\omega\colon f(\omega) \geq n\})$.

Solution

First we define appropriate $\mathcal{A}$-simple functions:

$$s_n \; := \; \sum_{k=1}^{n} k \, \mathbb{1}_{\{\omega\colon f(\omega)=k\}}.$$

Then $s_n \nearrow f$, which implies that

$$\int_\Omega f \, d\mu \; = \; \lim_{n\to\infty} \int_\Omega s_n \, d\mu \; = \; \lim_{n\to\infty} \sum_{k=1}^{n} k \, \mu(\{\omega\colon f(\omega) = k\}) \; = \; \sum_{k=1}^{\infty} k \, \mu(\{\omega\colon f(\omega) = k\}).$$

On the other hand, we have that

$$\begin{aligned} \sum_{n=1}^{\infty} \mu(\{\omega\colon f(\omega) \geq n\}) \; &= \; \sum_{n=1}^{\infty} \sum_{k \geq n} \mu(\{\omega\colon f(\omega) = k\}) \\ &= \; \sum_{k=1}^{\infty} \sum_{n=1}^{k} \mu(\{\omega\colon f(\omega) = k\}) \; = \; \sum_{k=1}^{\infty} k \, \mu(\{\omega\colon f(\omega) = k\}). \end{aligned}$$