**Measure Theory, Winter semester 2021/22**
Solutions to Problem sheet 6

12) Let $\mu$ and $\mu_n$ ($n \in \mathbb{N}$) be measures on a measurable space $(\Omega, \mathcal{A})$ such that $\mu_n(A) \nearrow \mu(A)$ for all $A \in \mathcal{A}$, and let $f\colon \Omega \to [0,\infty]$ be an $(\mathcal{A} - \bar{\mathcal{B}})$-measurable functions satisfying $\int_\Omega f\, d\mu < \infty$.

Show that

$$\int_\Omega f\, d\mu_n \nearrow \int_\Omega f\, d\mu.$$

*Hint: Show first that* $(\int_\Omega f\, d\mu_n)_{n\in\mathbb{N}}$ *is a non-decreasing sequence, and then that* $\lim_{n\to\infty} \int_\Omega f\, d\mu_n \leq \int_\Omega f\, d\mu$ *and* $\int_\Omega f\, d\mu \leq \lim_{n\to\infty} \int_\Omega f\, d\mu_n + \epsilon\ \forall \epsilon > 0$.

**Solution**

Let

$$\mathcal{S}_f := \{s\colon \Omega \to [0,\infty) \text{ is an } \mathcal{A}\text{-simple function, } s(\omega) \leq f(\omega)\ \forall \omega \in \Omega\}.$$

Then, for any $s = \sum_{i=1}^k \alpha_i \mathbb{1}_{A_i} \in \mathcal{S}_f$,

$$\int_\Omega s\, d\mu_n = \sum_{i=1}^k \alpha_i\, \mu_n(A_i) \nearrow \sum_{i=1}^k \alpha_i\, \mu(A_i) = \int_\Omega s\, d\mu,$$

which implies that

$$\int_\Omega f\, d\mu_n = \sup\Big\{ \int_\Omega s\, d\mu_n\colon s \in \mathcal{S}_f \Big\} \leq \sup\Big\{ \int_\Omega s\, d\mu_{n+1}\colon s \in \mathcal{S}_f \Big\} = \int_\Omega f\, d\mu_{n+1} \quad \forall n \in \mathbb{N}$$

as well as

$$\int_\Omega f\, d\mu_n = \sup\Big\{ \int_\Omega s\, d\mu_n\colon s \in \mathcal{S}_f \Big\} \leq \sup\Big\{ \int_\Omega s\, d\mu\colon s \in \mathcal{S}_f \Big\} = \int_\Omega f\, d\mu.$$

Hence, the limit of the integrals $\int_\Omega f\, d\mu_n$ exists and

$$\lim_{n\to\infty} \int_\Omega f\, d\mu_n \leq \int_\Omega f\, d\mu. \tag{1}$$

To prove the reverse inequality, choose any $\epsilon > 0$. Then there exists an $\mathcal{A}$-simple function $s = \sum_{i=1}^k \alpha_i \mathbb{1}_{A_i} \in \mathcal{S}_f$ such that

$$\int_\Omega f\, d\mu \leq \int_\Omega s\, d\mu + \epsilon.$$

On the other hand, we have that $\int_\Omega s\, d\mu_n \nearrow \int_\Omega s\, d\mu$, which implies that

$$\int_\Omega f\, d\mu \leq \lim_{n\to\infty} \int_\Omega s\, d\mu_n + \epsilon \leq \lim_{n\to\infty} \int_\Omega f\, d\mu_n + \epsilon. \tag{2}$$

(1) and (2) together imply that $\lim_{n\to\infty} \int_\Omega f\, d\mu_n = \int_\Omega f\, d\mu$.

13) Let $(\Omega, \mathcal{A})$ be a measurable space, let $\mu$ be an arbitrary measure on $(\Omega, \mathcal{A})$, and let $\nu$ be a finite measure on $(\Omega, \mathcal{A})$.

Show that $\nu \ll \mu$ if and only if for each $\epsilon > 0$ there is some $\delta = \delta(\epsilon) > 0$ such that each $\mathcal{A}$-measurable set $A$ that satisfies $\mu(A) < \delta$ also satisfies $\nu(A) < \epsilon$.

*Hint: For the proof that $\nu \ll \mu$ implies that for each $\epsilon$ there is a suitable $\delta$, assume that there exists some $\epsilon > 0$ and that there exist sets $A_k \in \mathcal{A}$ satisfying $\mu(A_k) < 1/2^k$ and $\nu(A_k) \geq \epsilon$. Show then that $\mu(\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k) = 0$ and $\nu(\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k) \geq \epsilon$.*

**Solution**

($\Longrightarrow$) Suppose that $\nu \ll \mu$.

We prove the conclusion by contradiction. Assume that there exists some $\epsilon > 0$ such that there exist $\mathcal{A}$-measurable sets $A_k$ satisfying $\mu(A_k) < 1/2^k$ and $\nu(A_k) \geq \epsilon$. Then $\mu(\bigcup_{k=n}^\infty A_k) \leq \sum_{k=n}^\infty \mu(A_k) < 2^{-n+1}$. Since $\mu(\bigcup_{k=n}^\infty A_k)$ is finite for all $n$ it follows by continuity from above that

$$\mu\Big(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\Big) = \lim_{n\to\infty} \mu\Big(\bigcup_{k=n}^{\infty} A_k\Big) = 0. \tag{3}$$

On the other hand, we have that $\nu(\bigcup_{k=n}^\infty A_k) \geq \nu(A_n) \geq \epsilon$ for all $n$. Since $\nu$ is a finite measure we have that $\nu(\bigcup_{k=n}^\infty A_k) < \infty$ and it follows again by continuity from above that

$$\nu\Big(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\Big) = \lim_{n\to\infty} \nu\Big(\bigcup_{k=n}^{\infty} A_k\Big) \geq \epsilon. \tag{4}$$

(3) and (4) together contradict $\nu \ll \mu$ and our assumption that there exists some $\epsilon > 0$ for which there is no suitable $\delta$ must be wrong.

($\Longleftarrow$) Suppose that for each $\epsilon > 0$ there exists some $\delta = \delta(\epsilon) > 0$ such that each $\mathcal{A}$-measurable set $A$ that satisfies $\mu(A) < \delta$ also satisfies $\nu(A) < \epsilon$.

Suppose now that $A \in \mathcal{A}$ and $\mu(A) = 0$. Since $\mu(A) < \delta(1/2^k)$ for all $k \in \mathbb{N}$ it follows that $\nu(A) < 1/2^k$, and so $\nu(A) = 0$. Hence, $\nu \ll \mu$.