Measure Theory, Winter semester 2021/22
Solutions to Problem sheet 7

14) The assumption of σ-finiteness in Theorem 2.6.2 is essential:
Consider the measure spaces $(\mathbb{R}, \mathcal{B}, \lambda)$ and $(\mathbb{R}, \mathcal{B}, \nu)$, where λ is Lebesgue measure and ν is counting measure.

(i) Show that the set $E := \{(\omega_1, \omega_2) \in \mathbb{R}^2\colon\ \omega_1 = \omega_2\}$ belongs to $\mathcal{B} \otimes \mathcal{B}$.

(ii) Compute $\int_{\mathbb{R}} \nu(E_{\omega_1})\, d\lambda(\omega_1)$ and $\int_{\mathbb{R}} \lambda(E^{\omega_2})\, d\nu(\omega_2)$.

Solution

(i) Let

$$E_n \;=\; \bigcup_{k=-\infty}^{\infty} \Big(\frac{k-1}{n}, \frac{k}{n}\Big] \times \Big(\frac{k-1}{n}, \frac{k}{n}\Big].$$

Since $(\frac{k-1}{n}] \times (\frac{k-1}{n}] \in \mathcal{B} \otimes \mathcal{B}$ it follows that $E_n \in \mathcal{B} \otimes \mathcal{B}$. From

$$E \;=\; \bigcap_{n=1}^{\infty} E_n$$

we obtain that $E \in \mathcal{B} \otimes \mathcal{B}$.

(ii) For each $\omega_1 \in \mathbb{R}$, $E_{\omega_1} = \{\omega_1\}$, and so

$$\nu(E_{\omega_1}) \;=\; 1.$$

This implies that

$$\int_{\mathbb{R}} \nu(E_{\omega_1})\, d\lambda(\omega_1) \;=\; \infty.$$

On the other hand, we have for each $\omega_2 \in \mathbb{R}$ that $E^{\omega_2} = \{\omega_2\}$, which implies that

$$\lambda(E^{\omega_2}) \;=\; 0.$$

Hence,

$$\int_{\mathbb{R}} \lambda(E^{\omega_2})\, d\nu(\omega_2) \;=\; 0.$$

15) Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, and let $f\colon \Omega \to [0,\infty]$ be a non-negative $(\mathcal{A}-\bar{\mathcal{B}})$-measurable function.

(i) Show that the set $E_f := \{(\omega, y) \in \Omega \times \mathbb{R}\colon\ 0 \le y < f(\omega)\}$ belongs to $\mathcal{A} \otimes \bar{\mathcal{B}}$.

(ii) Prove that

$$\int_\Omega f\, d\mu \;=\; (\mu \otimes \lambda)(E_f).$$

(The set E_f is the "area under the curve" and $(\mu \otimes \lambda)(E_f)$ is an alternative definition of the integral.)

Solution

(i) Let $s = \sum_{i=1}^k \alpha_i \mathbb{1}_{A_i}$ be an $\mathcal{A}$-simple function, i.e. $\alpha_1, \ldots \alpha_k \ge 0$ and $A_1, \ldots, A_k$ are disjoint sets that belong to $\mathcal{A}$. Then

$$\begin{aligned} E_s \;&:=\; \{(\omega, y) \in \Omega \times \mathbb{R}\colon\ 0 \le y < s(\omega)\} \\ &=\; \bigcup_{i=1}^k A_i \times [0, \alpha_i) \;\in\; \mathcal{A} \otimes \bar{\mathcal{B}}. \end{aligned}$$

For an arbitrary non-negative and $(\mathcal{A}-\bar{\mathcal{B}})$-measurable function $f\colon \Omega \to [0,\infty]$, there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of $\mathcal{A}$-simple functions such that

$$s_n(\omega) \nearrow f(\omega) \qquad \forall \omega \in \Omega.$$

Then

$$E_{s_n} \nearrow E_f,$$

i.e. $E_f = \bigcup_{n=1}^\infty E_{s_n} \in \mathcal{A} \otimes \bar{\mathcal{B}}$.

(ii) We have, for the above $\mathcal{A}$-simple function s,

$$\begin{aligned} \int_\Omega s\, d\mu \;&=\; \sum_{i=1}^k \alpha_i\, \mu(A_i) \;=\; \sum_{i=1}^k \lambda([0, \alpha_i)) \cdot \mu(A_i) \\ &=\; \sum_{i=1}^k (\mu \otimes \lambda)(A_i \times [0, \alpha_i)) \;=\; (\mu \otimes \lambda)(E_s). \end{aligned}$$

We obtain by Beppo Levi's theorem and continuity from below

$$\int_\Omega f\, d\mu \;=\; \lim_{n\to} \int_\Omega s_n\, d\mu \;=\; \lim_{n\to\infty} (\mu \otimes \lambda)(E_{s_n}) \;=\; (\mu \otimes \lambda)(E_f).$$

16) Let λ be Lebesgue measure on $(\mathbb{R}, \mathcal{B})$.

Compute $\int_{[0,1]} \left[\int_{[x,1]} e^{-y^2/2} \, d\lambda(y) \right] d\lambda(x)$.

Hint: Use the fact that $(e^{-y^2/2})' = -ye^{-y^2/2}$.

Solution

The function $\binom{x}{y} \mapsto e^{-y^2/2} \mathbb{1}_{[x,1]}(y)$ is non-negative and $(\mathcal{B}^2 - \mathcal{B})$-measurable. Therefore, we can apply Tonelli's theorem and we obtain that

$$\begin{aligned}
\int_{[0,1]} \left[\int_{[x,1]} e^{-y^2/2} \, d\lambda(y) \right] d\lambda(x) &= \int_{[0,1]\times[0,1]} e^{-y^2/2} \mathbb{1}_{[x,1]}(y) \, \lambda^2(dx, dy) \\
&= \int_{[0,1]} \left[\int_{[0,1]} e^{-y^2/2} \underbrace{\mathbb{1}_{[x,1]}(y)}_{=\mathbb{1}_{[0,y]}(x)} \, d\lambda(x) \right] d\lambda(y) \\
&= \int_{[0,1]} \Bigg[\underbrace{\int_{[0,1]} e^{-y^2/2} \mathbb{1}_{[0,y]}(x) \, d\lambda(x)}_{=e^{-y^2/2}y} \Bigg] d\lambda(y) \\
&= \int_0^1 \underbrace{e^{-y^2/2} y}_{=-(e^{-y^2/2})'} \; dy \;=\; 1 \;-\; 1/e.
\end{aligned}$$