Measure Theory, Winter semester 2021/22
Problem sheet 1

1) Let Ω be a non-empty set and let $\mathcal{A}$ be a σ-algebra on Ω. Suppose that P_1 and P_2 are probability measures on $(\Omega, \mathcal{A})$, i.e. $P_i \colon \mathcal{A} \to [0,1]$ satisfies

   (i) $P_i(\emptyset) = 0$, $P_i(\Omega) = 1$,

   (ii) if $A_1, A_2, \ldots$ are disjoint sets that belong to $\mathcal{A}$, then

$$P_i\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} P_i(A_i).$$

   Show that the collection of sets $\mathcal{D} := \{A \in \mathcal{A}\colon P_1(A) = P_2(A)\}$ is a Dynkin system on Ω.
   Hint: Note that $P_i(A^c) = P_i(\Omega \setminus A) = P_i(\Omega) - P_i(A)$.

2) Let Ω be a non-empty set and let $(\mathcal{A}_i)_{i \in I}$ be a non-empty collection of σ-algebras on Ω, where I is an arbitrary (finite, countably infinite or even uncountable) index set.

   Show that the intersection of these σ-algebras,

$$\bigcap_{i \in I} \mathcal{A}_i = \{A \subseteq \Omega\colon \quad A \in \mathcal{A}_i \quad \text{for all } i \in I\},$$

   is a σ-algebra on Ω.