Measure Theory, Winter semester 2021/22
Problem sheet 2

3) Let $\Omega = \mathbb{N}$ and let $\mathcal{C} := \{A \subseteq \Omega\colon\ A \text{ or } A^c \text{ is a finite subset of } \Omega\}$.

   (i) Show that $\mathcal{C}$ is a ring but not a σ-algebra on Ω.

   (ii) For $A \in \mathcal{C}$, let $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if A is infinite.
   Show that μ is a content on $(\Omega, \mathcal{C})$. Is μ a pre-measure on $(\Omega, \mathcal{C})$?

4) Let $P\colon \mathcal{B}^d \to [0,1]$ be a probability measure on $(\mathbb{R}^d, \mathcal{B}^d)$, and let $F\colon \mathbb{R}^d \to [0,1]$ be the corresponding distribution function, i.e.

$$F(x_1, \ldots, x_d) \; = \; P((-\infty, x_1] \times \cdots \times (-\infty, x_d]) \qquad \forall x_1, \ldots, x_d \in \mathbb{R}.$$

Show that, for all $x_1, \ldots, x_d \in \mathbb{R}$, $y_1, \ldots, y_d \geq 0$,

$$P((x_1, x_1{+}y_1] \times \cdots \times (x_d, x_d{+}y_d]) \; = \sum_{(\theta_1,\ldots,\theta_d)\in\{0,1\}^d} (-1)^{\sum_{i=1}^d (1-\theta_i)} F(x_1{+}\theta_1 y_1, \ldots, x_d{+}\theta_d y_d).$$

Hint: Consider the sets $A_i = (-\infty, x_1 + y_1] \times \cdots \times (-\infty, x_{i-1} + y_{i-1}] \times (-\infty, x_i] \times (-\infty, x_{i+1} + y_{i+1}] \times \cdots \times (-\infty, x_d + y_d]$.