Measure Theory, Winter semester 2021/22
Problem sheet 3

5) Let $\lambda^*\colon 2^{\mathbb{R}} \to [0,\infty]$ be the outer measure on $\mathbb{R}$ defined by

$$\lambda^*(Q) \;=\; \begin{cases} \{\sum_{i=1}^{\infty} \lambda_0^1(A_i)\colon \quad A_1, A_2, \ldots \in \mathcal{B}_0^1,\; Q \subseteq \bigcup_{i=1}^{\infty} A_i\} & \text{if } \{\ldots\} \neq \emptyset, \\ +\infty & \text{if } \{\ldots\} = \emptyset. \end{cases}$$

Using **only the definition** of λ^*, show that $\lambda^*(C) = 0$ if C is a countable subset of $\mathbb{R}$.

6) *(A model for countably many fair coin tosses)*
Let

$$\Omega \;=\; \{\omega = (\omega_1, \omega_2, \ldots)\colon \quad \omega_i \in \{0,1\} \text{ for all } i \in \mathbb{N}\},$$
$$\mathcal{C}_n \;=\; \{A \times \Omega\colon \quad A \subseteq \{0,1\}^n\} \;=\; \Big\{\{\omega \in \Omega\colon \quad (\omega_1, \ldots .\omega_n) \in A\}\Big|\; A \subseteq \{0,1\}^n\Big\}$$

and

$$\mathcal{C} \;=\; \bigcup_{n=1}^{\infty} \mathcal{C}_n.$$

Define $P_0\colon \mathcal{C} \to [0,1]$ such that, for $A \in \{0,1\}^n$ and $n \in \mathbb{N}$,

$$P_0(A \times \Omega) \;=\; P_0(\{\omega \in \Omega\colon (\omega_1, \ldots, \omega_n) \in A\}) \;=\; \frac{\#A}{2^n}.$$

(i) Show that $\mathcal{C}$ is an algebra (and therefore a ring) on Ω.

(ii) For each $c \in [0,1]$, let $B_c = \{\omega \in \Omega\colon \frac{1}{n}\sum_{i=1}^{n} \omega_i \underset{n\to\infty}{\longrightarrow} c\}$.
Show that $B_c \in \sigma(\mathcal{C})$.
(*Hint: Use that* $\{\omega \in \Omega\colon |\frac{1}{n}\sum_{i=1}^{n} \omega_i \;-\; c| \;\leq\; \frac{1}{m}\} \in \mathcal{C}$.)

(iii) Let $A \times \Omega = B \times \Omega$, where $A \subseteq \{0,1\}^m$ and $B \subseteq \{0,1\}^n$.
Show that

$$P_0(A \times \Omega) \;=\; P_0(B \times \Omega).$$

(iv) Show that P_0 is a content on $\mathcal{C}$.
(*Actually, it can also be shown that* P_0 *is a pre-measure on* $\mathcal{C}$.)