Measure Theory, Winter semester 2021/22
Problem sheet 6

12) Let μ and μ_n ($n \in \mathbb{N}$) be measures on a measurable space $(\Omega, \mathcal{A})$ such that $\mu_n(A) \nearrow \mu(A)$ for all $A \in \mathcal{A}$, and let $f\colon \Omega \to [0, \infty]$ be an $(\mathcal{A} - \bar{\mathcal{B}})$-measurable functions satisfying $\int_\Omega f\, d\mu < \infty$.

    Show that
$$\int_\Omega f\, d\mu_n \nearrow \int_\Omega f\, d\mu.$$

    Hint: Show first that $(\int_\Omega f\, d\mu_n)_{n\in\mathbb{N}}$ *is a non-decreasing sequence, and then that* $\lim_{n\to\infty} \int_\Omega f\, d\mu_n \le \int_\Omega f\, d\mu$ *and* $\int_\Omega f\, d\mu \le \lim_{n\to\infty} \int_\Omega f\, d\mu_n + \epsilon \ \forall \epsilon > 0$.

13) Let $(\Omega, \mathcal{A})$ be a measurable space, let μ be an arbitrary measure on $(\Omega, \mathcal{A})$, and let ν be a finite measure on $(\Omega, \mathcal{A})$.

    Show that $\nu \ll \mu$ if and only if for each $\epsilon > 0$ there is some $\delta = \delta(\epsilon) > 0$ such that each $\mathcal{A}$-measurable set A that satisfies $\mu(A) < \delta$ also satisfies $\nu(A) < \epsilon$.

    Hint: For the proof that $\nu \ll \mu$ *implies that for each* ϵ *there is a suitable* δ*, assume that there exists some* $\epsilon > 0$ *and that there exist sets* $A_k \in \mathcal{A}$ *satisfying* $\mu(A_k) < 1/2^k$ *and* $\nu(A_k) \ge \epsilon$*. Show then that* $\mu(\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k) = 0$ *and* $\nu(\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k) \ge \epsilon$.