Measure Theory, Winter semester 2021/22
Problem sheet 7

14) The assumption of σ-finiteness in Theorem 2.6.2 is essential:
Consider the measure spaces $(\mathbb{R}, \mathcal{B}, \lambda)$ and $(\mathbb{R}, \mathcal{B}, \nu)$, where λ is Lebesgue measure and ν is counting measure.

   (i) Show that the set $E := \{(\omega_1, \omega_2) \in \mathbb{R}^2 \colon \omega_1 = \omega_2\}$ belongs to $\mathcal{B} \otimes \mathcal{B}$.

   (ii) Compute $\int_{\mathbb{R}} \nu(E_{\omega_1})\, d\lambda(\omega_1)$ and $\int_{\mathbb{R}} \lambda(E^{\omega_2})\, d\nu(\omega_2)$.

15) Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, and let $f \colon \Omega \to [0, \infty]$ be a non-negative $(\mathcal{A} - \bar{\mathcal{B}})$-measurable function.

   (i) Show that the set $E_f := \{(\omega, y) \in \Omega \times \mathbb{R} \colon 0 \leq y < f(\omega)\}$ belongs to $\mathcal{A} \otimes \bar{\mathcal{B}}$.

   (ii) Prove that
$$\int_\Omega f\, d\mu \;=\; (\mu \otimes \lambda)(E_f).$$
   (The set E_f is the "area under the curve" and $(\mu \otimes \lambda)(E_f)$ is an alternative definition of the integral.)

16) Let λ be Lebesgue measure on $(\mathbb{R}, \mathcal{B})$.

Compute $\int_{[0,1]} \Big[\int_{[x,1]} e^{-y^2/2}\, d\lambda(y) \Big]\, d\lambda(x)$.

Hint: Use the fact that $(e^{-y^2/2})' = -y e^{-y^2/2}$.