Mathematical Statistics, Winter semester 2021/22
Solutions to Problem sheet 4

10) Let $X \sim P_\theta = \text{Poisson}(\theta)$, where $\theta \in \Theta = (0, \infty)$.

(i) Compute the Fisher information of the family $\{P_\theta \colon \theta \in \Theta\}$.

(ii) Compute the mean squared error of the estimator $T(X) = X$ for the parameter θ.

Hint: Compute first $E_\theta X$ and $E_\theta[X(X-1)]$.

Solution

We compute the first two moments of $X \sim \text{Poisson}(\theta)$:

$$E_\theta X = \sum_{k=0}^{\infty} k e^{-\theta} \frac{\theta^k}{k!} = \theta \sum_{k=1}^{\infty} e^{-\theta} \frac{\theta^{k-1}}{(k-1)!} = \theta$$

and, since

$$E_\theta\big[X(X-1)\big] = \sum_{k=0}^{\infty} k(k-1) e^{-\theta} \frac{\theta^k}{k!} = \theta^2 \sum_{k=2}^{\infty} e^{-\theta} \frac{\theta^{k-2}}{(k-2)!} = \theta^2,$$

we obtain

$$E_\theta[X^2] = E_\theta\big[X(X-1)\big] + E_\theta X = \theta^2 + \theta.$$

This implies that

$$\text{var}_\theta(X) = E_\theta[X^2] - (E_\theta X)^2 = \theta.$$

Now we compute the Fisher information: For $p_\theta(k) = P_\theta(X = k) = e^{-\theta}\theta^k/k!$, we have

$$\frac{d}{d\theta} p_\theta(k) = -e^{-\theta} \frac{\theta^k}{k!} + e^{-\theta} \frac{k\theta^{k-1}}{k!} = (\frac{k}{\theta} - 1) e^{-\theta} \frac{\theta^k}{k!},$$

which leads to a score function l_θ such that

$$l_\theta(k) = (\frac{k}{\theta} - 1) \qquad \forall k = 0, 1, \ldots$$

Therefore,

$$I(\theta) = E_\theta[(l_\theta(X))^2] = E_\theta\Big[\Big(\frac{X}{\theta} - 1\Big)^2\Big] = \frac{1}{\theta}.$$

MSE:

$$E_\theta\Big[(X - \theta)^2\Big] = \theta = \frac{1}{I(\theta)}$$

$\Longrightarrow$ X is BUE

11) Compute the Fisher information number $I(\theta)$ of the family $\{N(\theta, \sigma^2)\colon\ \theta \in \mathbb{R}\}$. ($\sigma^2 > 0$ is fixed.)

Solution

The normal distribution with location parameter θ and variance σ^2 has a density p_θ w.r.t. the Lebesgue measure λ, where

$$p_\theta(x) \;=\; \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\theta)^2}{2\sigma^2}}.$$

Since

$$\frac{d}{d\theta} p_\theta(x) \;=\; \frac{x-\theta}{\sigma^2}\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\theta)^2}{2\sigma^2}} \;=\; \frac{x-\theta}{\sigma^2} p_\theta(x)$$

we obtain a score function l_θ such that

$$l_\theta(x) \;=\; \frac{x-\theta}{\sigma^2}.$$

The Fisher information at the point θ is given by

$$I(\theta) \;=\; E_\theta[l_\theta^2(X)] \;=\; E_\theta\Big[\frac{(X-\theta)^2}{\sigma^4}\Big] \;=\; \frac{1}{\sigma^2} \qquad \forall \theta \in \mathbb{R}.$$

It can also be shown that the family $\{N(\theta, \sigma^2)\colon\ \theta \in \mathbb{R}\}$ satisfies the regularity conditions **(C1)** and **(C2)**.

12) Let $X_1, \ldots, X_n$ be independent random variables, $X_i \sim \text{Uniform}[\theta_1, \theta_2]$, where $\theta = \binom{\theta_1}{\theta_2} \in \Theta := \{\binom{a}{b} \colon -\infty < a < b < \infty\}$.

Show that $T(X_1, \ldots, X_n) = (X_{n:1}, X_{n:n})$, where $X_{n:1} = \min\{X_1, \ldots, X_n\}$ and $X_{n:n} = \max\{X_1, \ldots, X_n\}$, is a sufficient statistic for $\theta \in \Theta$.

Solution