Mathematical Statistics, Winter semester 2021/22
Solutions to Problem sheet 5

13) Let $X_1, \dots, X_n$ be i.i.d. with $P_\theta(X_i = 1) = \theta = 1 - P_\theta(X_i = 0)$, where $\theta \in \Theta = (0, 1)$. Show with the aid of Proposition 2.9 (Lecture notes, page 46) that $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ is admissible (w.r.t. the mean squared error) in the class of all estimators.

Solution
The family $\{P_\theta^{(X_1,\dots,X_n)} \colon \theta \in (0,1)\}$ has the Fisher information

$$I(\theta) \;=\; \frac{n}{\theta(1-\theta)} \qquad \forall \theta \in (0,1);$$

see page 41 in the Lecture notes. The estimator $\bar{X}_n$ is unbiased for θ and it holds that

$$E_\theta[(\bar{X}_n - \theta)^2] \;=\; \frac{\theta(1-\theta)}{n} \;=\; \frac{1}{I(\theta)} \qquad \forall \theta \in (0,1).$$

(Therefore, $\bar{X}_n$ is a best unbiased estimator of θ.)
Since

$$\int_0^\theta I(u)\, du \;=\; \int_\theta^1 I(u)\, du \;=\; \infty \qquad \forall \theta \in (0,1)$$

we obtain from Proposition 2.9 that $\bar{X}_n$ is admissible in the class of **all** estimators.

14) Suppose that a realization of $X \sim P_\theta := \text{Bin}(\theta, p)$ is observed, where $\theta \in \Theta := \mathbb{N}$ and $p \in (0,1)$ is known. Let $\pi = \text{Poisson}(\lambda)$, $\lambda > 0$, be the prior distribution for θ.

(i) Find the posterior distribution of θ given $X = k$.

(ii) Suppose that the mean squared error is chosen as measure of the performance of an estimator. Compute the Bayes estimator.

Solution

(i) P_t and π have respective densities $p_{X|\theta=t}$ and p_θ w.r.t. the counting measures on $\mathbb{N}_0$ and $\mathbb{N}$, respectively, where

$$\begin{aligned} p_{X|\theta=t}(k) &= \binom{t}{k} p^k (1-p)^{t-k} \qquad (k = 0, 1, \ldots, \theta), \\ p_\theta(t) &= e^{-\lambda} \frac{\lambda^t}{t!} \qquad (t \in \mathbb{N}). \end{aligned}$$

Hence, the joint distribution of X and θ has a density $p_{X,\theta}$ w.r.t. the counting measure on $\mathbb{N}_0 \times \mathbb{N}$, where

$$p_{X,\theta}(k,t) \;=\; p_{X|\theta=t}(k)\, p_\theta(\theta) \;=\; \binom{t}{k} p^k (1-p)^{t-k}\, e^{-\lambda} \frac{\lambda^t}{t!}.$$

(Note that $\binom{t}{k} = 0$ if $k > t$.) To determine the posterior distribution of θ given $X = k$, we use the fact that the joint density of X and θ can also be written as

$$p_{X,\theta}(k,t) \;=\; p_X(k)\, p_{\theta|X=k}(t),$$

where p_X denotes the (unconditional) density of X. To this end, we separate the terms in $p_{X,\theta}(k,t)$ which contain the parameter t from those without t:

$$\begin{aligned} p_{X,\theta}(k,t) &= e^{-\lambda} \frac{p^k}{k!} \; \frac{(1-p)^{t-k}\, \lambda^t}{(t-k)!} \mathbb{1}(t \geq k) \\ &= \underbrace{e^{-\lambda p} \frac{(\lambda p)^k}{k!}}_{=: p_X(k)} \; \underbrace{e^{-\lambda(1-p)} \frac{(\lambda(1-p))^{t-k}}{(t-k)!} \mathbb{1}(t \geq k)}_{=: p_{\theta|X=k}(t)}. \end{aligned}$$

We see from this formula that the prior distribution of θ given $X = k$ is equal to that of $Y + k$, where $Y \sim \text{Poisson}(\lambda(1-p))$.

(ii) It follows from (i) that the Bayes estimator $T^* = T^*(X)$ is given by

$$T^*(k) \;=\; E(\theta \mid X = k) \;=\; E[Y + k] \;=\; \lambda(1-p) + k.$$

Note that $T^*(X)$ of θ is not integer-valued if $\lambda(1-p)$ is not an integer.
In this case, it makes sense to seek an estimator $T^{**}(X)$ which minimizes the Bayes risk in the class of all **integer-valued** estimators.
The Bayes risk of T^* can be written as

$$r(T^*, \pi) \;=\; \sum_{k=0}^{\infty} E((T^*(k) - \theta)^2 | X = k)\, p_X(k).$$

For an arbitrary (integer-valued) estimator $T^{**}(X)$ we obtain that

$$\begin{aligned} r(T^{**}, \pi) \;&=\; \sum_{k=0}^{\infty} E((T^{**}(k) - \theta)^2 | X = k)\, p_X(k) \\ &=\; \sum_{k=0}^{\infty} \Big\{ E((T^*(k) - \theta)^2 | X = k) \;+\; (T^{**}(k) - T^*(k))^2 \\ &\qquad + \underbrace{2\, E((T^{**}(k) - T^*(k))(T^*(k) - \theta) | X = k)}_{=\,2(T^{**}(k) - T^*(k))\, E(T^*(k) - \theta | X = x)\, =\, 0} \Big\}\, p_X(k). \end{aligned}$$

Hence, the sought integer-valued estimator is given by $T^{**}(X) = c + X$, where c is the integer closest to $\lambda(1-p)$.