Mathematical Statistics, Winter semester 2021/22
Solutions to Problem sheet 6

15) Let $X_1, \ldots, X_n$ be i.i.d. with $X_i \sim \mathrm{Bin}(1, \theta)$, where $\theta \in \Theta := \{\theta_0, \theta_1\} \subseteq (0,1)$, $\theta_0 \neq \theta_1$. For $\beta \in [0,1]$, find a (possibly randomized) test φ which minimizes

$$\beta\ E_{\theta_0}[\varphi(X)] \ + \ (1-\beta)\ E_{\theta_1}[1-\varphi(X)].$$

$(X = (X_1, \ldots, X_n)^T)$

Solution
Let $X = (X_1, \ldots, X_n)^T$. Then, for a test $\varphi\colon \{0,1\}^n \to [0,1]$,

$$\begin{aligned}
&\beta\, E_{\theta_0}[\varphi(X)] \ + \ (1-\beta)\, E_{\theta_1}[1 \ - \ \varphi(X)] \\
&\quad = \ (1-\beta) \ + \ \beta\, E_{\theta_0}[\varphi(X)] \ - \ (1-\beta)\, E_{\theta_1}[\varphi(X)] \\
&\quad = \ (1-\beta) \ + \sum_{x\in\{0,1\}} \varphi(x) \Big\{ \beta\, \theta_0^{\sum_i x_i} (1-\theta_0)^{n-\sum_i x_i} \ - \ (1-\beta)\, \theta_1^{\sum_i x_i} (1-\theta_1)^{n-\sum_i x_i} \Big\}.
\end{aligned}$$

Therefore, an optimal test φ is given by

$$\varphi(x) \ = \ \begin{cases} 1, & \text{if } \ (1-\beta)\, \theta_1^{\sum_i x_i} (1-\theta_1)^{n-\sum_i x_i} \ > \ \beta\, \theta_0^{\sum_i x_i} (1-\theta_0)^{n-\sum_i x_i}, \\ 0, & \text{if } \ (1-\beta)\, \theta_1^{\sum_i x_i} (1-\theta_1)^{n-\sum_i x_i} \ < \ \beta\, \theta_0^{\sum_i x_i} (1-\theta_0)^{n-\sum_i x_i}, \\ \text{arbitrary}, & \text{if } \ (1-\beta)\, \theta_1^{\sum_i x_i} (1-\theta_1)^{n-\sum_i x_i} \ = \ \beta\, \theta_0^{\sum_i x_i} (1-\theta_0)^{n-\sum_i x_i}. \end{cases}$$

16) Let $X_1, \dots, X_n$ be independent random variables with $X_i \sim \mathcal{N}(\theta, 1)$, $i = 1, \dots, n$. Consider the problem of testing the following hypotheses.

$$H_0\colon \quad \theta = \theta_0 \qquad \text{vs.} \qquad H_1\colon \quad \theta = \theta_1,$$

where $\theta_0 < \theta_1$.

How large must the sample size n be in order that the probabilities of type I and type II errors are both not greater than 0.05?

Hint: It holds that $\Phi^{-1}(0.95) \approx 1.64$.

Solution

For any n, the most powerful test $\varphi_{0.05}$ has the form

$$\varphi_{0.05}(x) = \begin{cases} 1, & \text{if } \sqrt{n}(\bar{x}_n - \theta_0) \geq \Phi^{-1}(0.95) \approx 1.64, \\ 0, & \text{if } \sqrt{n}(\bar{x}_n - \theta_0) < \Phi^{-1}(0.95). \end{cases}$$

Now we consider the probability of a type II error:

$$\begin{aligned} E_{\theta_1}[1 - \varphi_{0.05}(X)] &= P_{\theta_1}(\sqrt{n}(\bar{X}_n - \theta_0) < 1.64) \\ &= P_{\theta_1}(\underbrace{\sqrt{n}(\bar{X}_n - \theta_1)}_{\sim N(0,1)} < 1.64 + \sqrt{n}(\theta_0 - \theta_1)). \end{aligned}$$

Hence,

$$\begin{aligned} & E_{\theta_1}[1 - \varphi_{0.05}(X)] \geq 0.05 \\ \iff \; & 1.64 + \sqrt{n}(\theta_0 - \theta_1) \leq -1.64 \\ \iff \; & 3.28 \leq \sqrt{n}(\theta_1 - \theta_0) \\ \iff \; & n \geq \left(\frac{3.28}{\theta_1 - \theta_0}\right)^2. \end{aligned}$$

17) (i) Show that the family of distributions $\{\text{Bin}(n,\theta)\colon \theta \in (0,1)\}$ has a monotone likelihood ratio.

(ii) For $X \sim \text{Bin}(n,\theta)$, construct a UMP test of size $\alpha \in (0,1)$ for the problem

$$H_0\colon \quad \theta \leq 1/2 \qquad \text{vs.} \qquad H_1\colon \quad \theta > 1/2.$$

Solution

(i) Let $\theta_1, \theta_2 \in (0,1)$ be arbitrary such that $\theta_1 < \theta_2$. Then, for $k = 0, 1, \ldots, n$,

$$\frac{p_{\theta_2}(k)}{p_{\theta_1}(k)} = \frac{\binom{n}{k}\theta_2^k(1-\theta_2)^{n-k}}{\binom{n}{k}\theta_1^k(1-\theta_1)^{n-k}} = \left(\frac{1-\theta_2}{1-\theta_1}\right)^n \Bigg(\underbrace{\frac{\theta_2(1-\theta_1)}{\theta_1(1-\theta_2)}}_{>1}\Bigg)^k.$$

Therefore, the mapping $k \mapsto p_{\theta_2}(k)/p_{\theta_1}(k)$ is strictly monotonically increasing.

(ii) According to Theorem 3.3, a most powerful level α test φ_α is given by

$$\varphi_\alpha(k) = \begin{cases} 1, & \text{if } k > c_\alpha, \\ \gamma_\alpha, & \text{if } k = c_\alpha, \\ 0, & \text{if } k < c_\alpha, \end{cases}$$

where $c_\alpha \in \mathbb{N}_0$ and $\gamma_\alpha \in [0,1]$ are chosen such that

$$E_{\theta_0}\varphi_\alpha(X) = P_{\theta_0}(X > c_\alpha) + \gamma_\alpha\, P_{\theta_0}(X = c_\alpha) = \alpha.$$