Mathematical Statistics, Winter semester 2021/22
Solutions to Problem sheet 7

18) Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be independent random variables, where $X_i \sim \mathcal{N}(\theta_1, 1)$ and $Y_i \sim \mathcal{N}(\theta_2, 1)$.

Find a likelihood ratio test of size $\alpha > 0$ for

$$H_0\colon \quad \theta_1 = \theta_2 \qquad \text{vs.} \qquad H_1\colon \quad \theta_1 \neq \theta_2.$$

Solution

Let $X := (X_1, \ldots, X_n, Y_1, \ldots, Y_n)^T$. Then $X \sim \mathcal{N}\Big(\binom{\theta_1 \mathbb{1}_n}{\theta_2 \mathbb{1}_n}, I_{2n}\Big)$.

Testing problem: For $\theta := \binom{\theta_1}{\theta_2}$,

$$H_0\colon\ \theta \in \Theta_0 := \left\{ \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \colon\ \alpha \in \mathbb{R} \right\} \quad \text{vs.} \quad H_1\colon\ \theta \in \Theta_1 := \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \colon\ \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

Let $x := (x_1, \ldots, x_n, y_1, \ldots, y_n)^T$. Since $\theta \mapsto p_\theta(x)$ is continuous for all x, and Θ_1 is dense in $\Theta := \Theta_0 \cup \Theta_1$,

$$L(x) = \frac{\sup\{p_\theta(x)\colon\ \theta \in \Theta_1\}}{\sup\{p_\theta(x)\colon\ \theta \in \Theta_0\}} = \frac{\sup\{p_\theta(x)\colon\ \theta \in \Theta\}}{\sup\{p_\theta(x)\colon\ \theta \in \Theta_0\}} = \frac{p_{\widehat{\theta}}(x)}{p_{\widehat{\theta}_0}(x)} =: \widetilde{L}(x),$$

where $\widehat{\theta}$ and $\widehat{\theta}_0$ are respective maximum likelihood estimators in Θ and Θ_0,

$$\widehat{\theta} = \begin{pmatrix} \bar{X}_n \\ \bar{Y}_n \end{pmatrix}, \qquad \widehat{\theta}_0 = \begin{pmatrix} (\bar{X}_n + \bar{Y}_n)/2 \\ (\bar{X}_n + \bar{Y}_n)/2 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} &\widetilde{L}(x) \\ &\quad = \frac{p_{\widehat{\theta}}(x)}{p_{\widehat{\theta}_0}(x)} \\ &\quad = \frac{\exp\{ -\frac{1}{2}[\sum_i (x_i - \bar{x}_n)^2 \ + \ \sum_i (y_i - \bar{y}_n)^2]\}}{\exp\{ -\frac{1}{2}[\sum_i (x_i - \bar{x}_n + \bar{x}_n - (\bar{x}_n + \bar{y}_n)/2)^2 \ + \ \sum_i (y_i - \bar{y}_n + \bar{y}_n - (\bar{x}_n + \bar{y}_n)/2)^2]\}} \\ &\quad = \exp\left\{ \frac{n}{2}\Big[\underbrace{\Big(\bar{x}_n - \frac{\bar{x}_n + \bar{y}_n}{2}\Big)^2}_{= (\bar{x}_n - \bar{y}_n)^2/4} + \underbrace{\Big(\bar{y}_n - \frac{\bar{x}_n + \bar{y}_n}{2}\Big)^2}_{= (\bar{x}_n - \bar{y}_n)^2/4} \Big] \right\} \end{aligned}$$

$\Longrightarrow \quad \widetilde{L}(x)$ is monotonically increasing in $|\bar{x}_n - \bar{y}_n|$.

Since, under H_0, $\bar{X}_n - \bar{Y}_n \sim \mathcal{N}(0, 2/n)$, the size α likelihood ratio test φ_α is given by

$$\varphi_\alpha(x) = \begin{cases} 1, & \text{if } \sqrt{n/2}|\bar{x}_n - \bar{y}_n| \geq \Phi^{-1}(1 - \alpha/2), \\ 0, & \text{if } \sqrt{n/2}|\bar{x}_n - \bar{y}_n| < \Phi^{-1}(1 - \alpha/2). \end{cases}$$

19) Assume that Z has a t-distribution with n degrees of freedom.

Show that

$$P(Z \le t) \;=\; P(-Z \le t) \qquad \forall t \in \mathbb{R}.$$

Hint: Use the fact that, for $X \sim \mathcal{N}(0,1)$, $\;P(X \le u) = P(-X \le u)$ *holds for all* $u \in \mathbb{R}$.

Solution

Let $\widetilde{X}$ and $\widetilde{Y}$ be independent random variables which are defined on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{P})$ such that $\widetilde{X} \sim \mathcal{N}(0,1)$ and $\widetilde{Y} \sim \chi^2_n$. Then the random variable Z has the same distribution as $\widetilde{Z} := \widetilde{X}/\sqrt{\widetilde{Y}/n}$ has under $\widetilde{P}$.

We obtain that

$$\begin{aligned}
P(Z \le t) &= \widetilde{P}(\widetilde{Z} \le t) \\
&= \widetilde{P}\Big(\widetilde{X}/\sqrt{\widetilde{Y}/n} \le t\Big) \\
&= \int_{\mathbb{R}} \underbrace{\widetilde{P}\Big(\widetilde{X} \le t\sqrt{\widetilde{Y}/n}\,\Big|\,\widetilde{Y} = y\Big)}_{=\widetilde{P}(\widetilde{X} \le t\sqrt{y/n})} \, d\widetilde{P}^{\widetilde{Y}}(y) \\
&= \int_{\mathbb{R}} \underbrace{\widetilde{P}(\widetilde{X} \le t\sqrt{y/n})}_{=\widetilde{P}(-\widetilde{X} \le t\sqrt{y/n})} \, d\widetilde{P}^{\widetilde{Y}}(y) \\
&= \widetilde{P}\Big(-\widetilde{X}/\sqrt{\widetilde{Y}/n} \le t\Big) \;=\; P(-Z \le t).
\end{aligned}$$

20) Show that the one-sided t-test is unbiased.

Solution

Framework for a one-sided t test: $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ independent,

$$H_0\colon \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} \in (-\infty, \mu_0] \times (0, \infty) \qquad \text{vs.} \qquad H_1\colon \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} \in (\mu_0, \infty) \times (0, \infty)$$

If α is the prescribed level of significance, then the corresponding likelihood ratio test φ_α is given by

$$\varphi_\alpha(x) \;=\; \begin{cases} 1 & \text{if } T_n(x) \geq t_{n-1;1-\alpha}, \\ 0 & \text{if } T_n(x) < t_{n-1;1-\alpha} \end{cases},$$

where

$$T_n(x) \;=\; \frac{\sqrt{n}(\bar{x}_n - \mu_0)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x}_n)^2}}.$$

If $\mu = \mu_0$, then $T_n(X) \sim t_{n-1}$ and

$$\sup_{\mu\colon \mu \leq \mu_0} E_{\binom{\mu}{\sigma^2}} \varphi_\alpha(X) \;=\; E_{\binom{\mu_0}{\sigma^2}} \varphi_\alpha(X) \;=\; P_{\binom{\mu_0}{\sigma^2}}\big(\underbrace{T_n(X)}_{\sim t_{n-1}} \geq t_{n-1;1-\alpha}\big) \;=\; \alpha.$$

If $\mu > \mu_0$, then

$$\begin{aligned} E_{\binom{\mu}{\sigma^2}} \varphi_\alpha(X) \;&=\; P_{\binom{\mu}{\sigma^2}}(T_n(X) \geq t_{n-1;1-\alpha}) \\ &=\; P_{\binom{\mu}{\sigma^2}}\Big(\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \geq t_{n-1;1-\alpha}\Big) \\ &=\; P_{\binom{\mu}{\sigma^2}}\Big(\underbrace{\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2}}}_{\sim t_{n-1}} \geq t_{n-1;1-\alpha} + \underbrace{\frac{\sqrt{n}(\mu_0 - \mu)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2}}}_{<0}\Big) \\ &>\; \alpha. \end{aligned}$$