Mathematical Statistics, Winter semester 2021/22
Problem sheet 1

1) Suppose that

$$Y = X\theta + \varepsilon$$

holds for some $\theta \in \mathbb{R}^k$, where $E\varepsilon = 0_n$, $\mathrm{Cov}(\varepsilon) = \Sigma$, Σ being a regular matrix. Suppose further that the matrix X has full column rank k.

   (i) Show that $X^T\Sigma^{-1}X$ is a regular matrix and that

$$\widehat{\theta} = (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1}Y$$

   is an unbiased estimator of θ. Compute $E_\theta\left[(\widehat{\theta}-\theta)(\widehat{\theta}-\theta)^T\right]$.

   (ii) Let $\widetilde{\theta} = LY$ be an arbitrary unbiased estimator of θ.
   Show that

$$E_\theta\left[(\widetilde{\theta}-\theta)(\widetilde{\theta}-\theta)^T\right] - E_\theta\left[(\widehat{\theta}-\theta)(\widehat{\theta}-\theta)^T\right]$$

   is non-negative definite.
   Hint: A symmetric and positive definite $(n \times n)$-matrix M can be represented as $M = \sum_{i=1}^n \lambda_i e_i e_i^T$, where $\lambda_1, \ldots, \lambda_n$ are the (positive) eigenvalues and $e_1, \ldots, e_n$ are corresponding eigenvectors with $e_i^T e_j = 0$ for $i \neq j$. Then $M^{1/2} := \sum_{i=1}^n \sqrt{\lambda_i} e_i e_i^T$ and $M^{-1/2} := \sum_{i=1}^n (1/\sqrt{\lambda_i}) e_i e_i^T$.
   To prove (ii), use the fact that

$$\left(L\Sigma^{1/2} - (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1/2}\right)\left(L\Sigma^{1/2} - (X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1/2}\right)^T$$

   is a non-negative definite matrix.

2) (i) Let

$$X = \begin{pmatrix} 1 & v_1 & v_1^2 & \cdots & v_1^k \\ 1 & v_2 & v_2^2 & \cdots & v_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & v_n & v_n^2 & \cdots & v_n^k \end{pmatrix}.$$

   Prove that X^TX is regular if the set $\{v_1, \ldots, v_n\}$ contains at least $k+1$ different values.
   Hint: Choose $c = (c_1, \ldots, c_{k+1})^T \neq 0_{k+1} := (0, \ldots, 0)^T$ and use the fact that $c^TX^TXc = \sum_{i=1}^n \left(\sum_{j=1}^k c_{j+1} v_i^j\right)^2$.

(ii) Let

$$X = \begin{pmatrix} v_1 & v_1^2 & \cdots & v_1^k \\ v_2 & v_2^2 & \cdots & v_2^k \\ \vdots & \vdots & \ddots & \vdots \\ v_n & v_n^2 & \cdots & v_n^k \end{pmatrix}.$$

Prove that $X^T X$ is regular if the set $\{v_1, \ldots, v_n\}$ contains at least k different non-zero values.

Hint: Consider the matrix

$$\widetilde{X} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & v_1 & v_1^2 & \cdots & v_1^k \\ 1 & v_2 & v_2^2 & \cdots & v_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & v_n & v_n^2 & \cdots & v_n^k \end{pmatrix}.$$

3) Consider the linear regression model $Y_i = \theta_1 + x_i\theta_2 + \varepsilon_i$, $i = 1, \ldots, n$, where $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d. with $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$. Let $\widehat{\theta} = (\widehat{\theta}_1, \widehat{\theta}_2)^T$ be the least squares estimator of $\theta = (\theta_1, \theta_2)^T$.

(i) Suppose that $x_i \neq x_j$, for some (i, j).

Compute $E_\theta[(\widehat{\theta}_i - \theta_i)^2]$, for $i = 1, 2$.

Hint: The inverse of a regular matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is given by* $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(ii) Suppose that $x_1, \ldots, x_n$ can be chosen by an experimenter, where $x_i \in [-1, 1]$ and $n \geq 2$ is even.

Which choice of $x_1, \ldots, x_n$ minimizes $E_\theta[(\widehat{\theta}_i - \theta_i)^2]$? (Take into account that $x_1, \ldots, x_n$ have to be chosen such that $x_i \neq x_j$, for some (i, j); otherwise the least squares estimator is not uniquely defined.)