Mathematical Statistics, Winter semester 2021/22
Problem sheet 2

4) Suppose that

$$Y_i = \theta + \varepsilon_i, \qquad i = 1, \dots, n,$$

holds for some $\theta \in \Theta := \mathbb{R}$, where $\varepsilon_1, \dots, \varepsilon_n$ are independent random variables such that $E\varepsilon_i = 0$ and $\operatorname{var}(\varepsilon_i) = \sigma_i^2 > 0$ for $i = 1, \dots, n$.

Compute the best linear estimator of θ.

5) Consider the linear regression model

$$\underbrace{\begin{pmatrix} Y_{11} \\ \vdots \\ Y_{1n_1} \\ \vdots \\ Y_{k1} \\ \vdots \\ Y_{kn_k} \end{pmatrix}}_{=:Y} = \underbrace{\begin{pmatrix} \mathbb{1}_{n_1} & & \\ & \ddots & \\ & & \mathbb{1}_{n_k} \end{pmatrix}}_{=:X} \underbrace{\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}}_{=:\beta} + \underbrace{\begin{pmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1n_1} \\ \vdots \\ \varepsilon_{k1} \\ \vdots \\ \varepsilon_{kn_k} \end{pmatrix}}_{=:\varepsilon},$$

where $\beta_1, \dots, \beta_k$ are unknown parameters.

Compute the least squares estimator $\widehat{\beta}$ of β and compute $X\widehat{\beta}$.

6) Consider the model

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \qquad (1 \le i \le k,\ 1 \le j \le m).$$

Rewrite this model in vector/matrix form $Y = \bar{X}\theta + \varepsilon$, where $\theta = (\mu, \alpha_1, \dots, \alpha_k)^T$ is the unknown parameter.
What is the rank of the matrix $\bar{X}$?
Compute the least squares estimator of θ under the side condition $\sum_{i=1}^k \alpha_i = 0$.
Hint: Consider the linear model from exercise 5). Since

$$\{\bar{X}\theta \colon \theta \in \mathbb{R}^{k+1}\} = \{X\beta \colon \beta \in \mathbb{R}^k\},$$

$\widehat{\theta}$ *can be chosen such that* $\bar{X}\widehat{\theta} = X\widehat{\beta}$.