A note on the spaces of variable integrability and summability of Almeida and Hästö

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Abstract

We address an open problem posed recently by Almeida and Hästö in [1]. They defined the spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$ of variable integrability and summability and showed that $\|\cdot|\ell_{q(\cdot)}(L_{p(\cdot)})\|$ is a norm if q is constant almost everywhere or if $\text{ess-sup}_{x\in\mathbb{R}^n} 1/p(x)+1/q(x)\le 1$. Nevertheless, the natural conjecture (expressed also in [1]) is that the expression is a norm if $p(x),q(x)\ge 1$ almost everywhere. We show, that $\|\cdot|\ell_{q(\cdot)}(L_{p(\cdot)})\|$ is a norm, if $1\le q(x)\le p(x)$ for almost every $x\in\mathbb{R}^n$. Furthermore, we construct an example of $p(x)$ and $q(x)$ with $\min(p(x),q(x))\ge 1$ for every $x\in\mathbb{R}^n$ such that the triangle inequality does not hold for $\|\cdot|\ell_{q(\cdot)}(L_{p(\cdot)})\|$.

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1 Introduction

For the definition of the spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$ we follow closely [1]. Spaces of variable integrability $L_{p(\cdot)}$ and variable sequence spaces $\ell_{q(\cdot)}$ have first been considered in 1931 by Orlicz [5] but the modern development started with the paper [4]. We refer to [3] for an excellent overview of the vastly growing literature on the subject.

First of all we recall the definition of the variable Lebesgue spaces $L_{p(\cdot)}(\Omega)$, where Ω is a measurable subset of $\mathbb{R}^n$. A measurable function $p:\Omega\to(0,\infty]$ is called a variable exponent function if it is bounded away from zero. For a set $A\subset\Omega$ we denote $p_A^+=\text{ess-sup}_{x\in A}\,p(x)$ and $p_A^-=\text{ess-inf}_{x\in A}\,p(x)$; we use the abbreviations $p^+=p_\Omega^+$ and $p^-=p_\Omega^-$. The variable exponent Lebesgue space $L_{p(\cdot)}(\Omega)$ consists of all measurable functions f such that there exist an $\lambda>0$ such that the modular

$$\varrho_{L_{p(\cdot)}(\Omega)}(f/\lambda)=\int_\Omega \varphi_{p(x)}\left(\frac{|f(x)|}{\lambda}\right)dx$$

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is finite, where

$$\varphi_p(t) = \begin{cases} t^p & \text{if } p \in (0,\infty), \\ 0 & \text{if } p = \infty \text{ and } t \le 1, \\ \infty & \text{if } p = \infty \text{ and } t > 1. \end{cases}$$

This definition is nowadays standard and was used also in [1, Section 2.2] and [3, Definition 3.2.1].

If we define $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$ and $\Omega_0 = \Omega \setminus \Omega_\infty$, then the Luxemburg norm of a function $f \in L_{p(\cdot)}(\Omega)$ is given by

$$\begin{aligned} \big\| f \,|\, L_{p(\cdot)}(\Omega)\big\| &= \inf\{\lambda > 0 : \varrho_{L_{p(\cdot)}(\Omega)}(f/\lambda) \le 1\} \\ &= \inf\left\{\lambda > 0 : \int_{\Omega_0}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1 \text{ and } |f(x)| \le \lambda \text{ for a.e. } x \in \Omega_\infty\right\}. \end{aligned}$$

If $p(\cdot) \ge 1$, then it is a norm, but it is always a quasi-norm if at least $p^- > 0$, see [4] for details. We denote the class of all measurable functions $p : \mathbb{R}^n \to (0,\infty]$ such that $p^- > 0$ by $\mathcal{P}(\mathbb{R}^n)$.

To define the mixed spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$ we have to define another modular. For $p, q \in \mathcal{P}(\mathbb{R}^n)$ and a sequence $(f_\nu)_{\nu\in\mathbb{N}_0}$ of $L_{p(\cdot)}(\mathbb{R}^n)$ functions we define

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \inf\left\{\lambda_\nu > 0 : \varrho_{p(\cdot)}\left(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}}\right) \le 1\right\} . \tag{1}$$

The (quasi-) norm in the $\ell_{q(\cdot)}(L_{p(\cdot)})$ spaces is defined as usually by

$$\big\| f_\nu \,|\, \ell_{q(\cdot)}(L_{p(\cdot)})\big\| = \inf\{\mu > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu/\mu) \le 1\} . \tag{2}$$

This (quasi-) norm was used in [1] to define the spaces of Besov type with variable integrability and summability. Spaces of Triebel-Lizorkin type with variable indices have been considered recently in [2]. The appropriate $L_{p(\cdot)}(\ell_{q(\cdot)})$ space is a normed space whenever $\text{ess-inf}_{x\in\mathbb{R}^n} \min(p(x), q(x)) \ge 1$. This was the expected result and coincides with the case of constant exponents.

As pointed out in the remark after Theorem 3.8 in [1], the same question is still open for the $\ell_{q(\cdot)}(L_{p(\cdot)})$ spaces.

2 When does $\big\| \cdot \,|\, \ell_{q(\cdot)}(L_{p(\cdot)})\big\|$ define a norm?

In Theorem 3.6 of [1] the authors proved that if the condition $\frac{1}{p(x)} + \frac{1}{q(x)} \le 1$ holds for almost every $x \in \mathbb{R}^n$, then $\big\| \cdot \,|\, \ell_{q(\cdot)}(L_{p(\cdot)})\big\|$ defines a norm. They also proved in Theorem 3.8 that $\big\| \cdot \,|\, \ell_{q(\cdot)}(L_{p(\cdot)})\big\|$ is a quasi-norm for all $p, q \in \mathcal{P}(\mathbb{R}^n)$. Furthermore, the authors of [1] posed a question if the (rather natural) condition $p(x), q(x) \ge 1$ for almost every $x \in \mathbb{R}^n$ ensures that $\big\| \cdot \,|\, \ell_{q(\cdot)}(L_{p(\cdot)})\big\|$ is a norm.

We give (in Theorem 1) a positive answer if $1 \le q(x) \le p(x) \le \infty$ almost everywhere on $\mathbb{R}^n$. Furthermore in Theorem 2, we construct two functions $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, such that $\inf_{x\in\mathbb{R}^n} \min(p(x), q(x)) \ge 1$, but the triangle inequality does not hold for $\big\| \cdot \,|\, \ell_{q(\cdot)}(L_{p(\cdot)})\big\|$.

2.1 Positive results

Theorem 1. *Let $p, q \in \mathcal{P}(\mathbb{R}^n)$, such that either $1 \le q(x) \le p(x) \le \infty$ for almost every $x \in \mathbb{R}^n$ or $1/p(x) + 1/q(x) \le 1$ for almost every $x \in \mathbb{R}^n$. Then $\left\| \cdot | \, \ell_{q(\cdot)}(L_{p(\cdot)}) \right\|$ defines a norm.*

Proof. We want to show, that

$$\|f_\nu + g_\nu | \ell_{q(\cdot)}(L_{p(\cdot)})\| \le \|f_\nu | \ell_{q(\cdot)}(L_{p(\cdot)})\| + \|g_\nu | \ell_{q(\cdot)}(L_{p(\cdot)})\| \tag{3}$$

for all sequences of measurable functions $\{f_\nu\}_{\nu\in\mathbb{N}_0}$ and $\{g_\nu\}_{\nu\in\mathbb{N}_0}$. Let $\mu_1 > 0$ and $\mu_2 > 0$ be given with

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\left(\frac{f_\nu}{\mu_1}\right) \le 1 \quad \text{and} \quad \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\left(\frac{g_\nu}{\mu_2}\right) \le 1.$$

We want to show, that

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\left(\frac{f_\nu + g_\nu}{\mu_1 + \mu_2}\right) \le 1.$$

For every $\varepsilon > 0$, there exist sequences of positive numbers $\{\lambda_\nu\}_{\nu\in\mathbb{N}_0}$ and $\{\Lambda_\nu\}_{\nu\in\mathbb{N}_0}$, such that

$$\varrho_{p(\cdot)}\left(\frac{f_\nu(x)}{\mu_1 \lambda_\nu^{1/q(x)}}\right) \le 1 \quad \text{and} \quad \varrho_{p(\cdot)}\left(\frac{g_\nu(x)}{\mu_2 \Lambda_\nu^{1/q(x)}}\right) \le 1 \tag{4}$$

together with

$$\sum_{\nu=0}^{\infty} \lambda_\nu \le 1 + \varepsilon \quad \text{and} \quad \sum_{\nu=0}^{\infty} \Lambda_\nu \le 1 + \varepsilon.$$

We set

$$A_\nu := \frac{\mu_1 \lambda_\nu + \mu_2 \Lambda_\nu}{\mu_1 + \mu_2}, \quad \text{i.e.} \quad \sum_{\nu=0}^{\infty} A_\nu \le 1 + \varepsilon.$$

We shall prove, that

$$\varrho_{p(\cdot)}\left(\frac{f_\nu(x) + g_\nu(x)}{A_\nu^{1/q(x)}(\mu_1 + \mu_2)}\right) \le 1 \quad \text{for all} \quad \nu \in \mathbb{N}_0. \tag{5}$$

Let $\Omega_0 := \{x \in \mathbb{R}^n : p(x) < \infty\}$ and $\Omega_\infty := \{x \in \mathbb{R}^n : p(x) = \infty\}$. We put for every $x \in \Omega_0$

$$F_\nu(x) := \left(\frac{|f_\nu(x)|}{\mu_1 \lambda_\nu^{1/q(x)}}\right)^{p(x)} \quad \text{and} \quad G_\nu(x) := \left(\frac{|g_\nu(x)|}{\mu_2 \Lambda_\nu^{1/q(x)}}\right)^{p(x)}.$$

Then (4) may be reformulated as

$$\int_{\Omega_0} F_\nu(x) dx \le 1 \quad \text{and} \quad \operatorname*{ess-sup}_{x\in\Omega_\infty} \frac{|f_\nu(x)|}{\mu_1 \lambda_\nu^{1/q(x)}} \le 1 \tag{6}$$

and

$$\int_{\Omega_0} G_\nu(x) dx \le 1 \quad \text{and} \quad \operatorname*{ess-sup}_{x\in\Omega_\infty} \frac{|g_\nu(x)|}{\mu_2 \Lambda_\nu^{1/q(x)}} \le 1 \, . \tag{7}$$

Our aim is to prove (5), which reads

$$\int_{\Omega_0}\left(\frac{|f_\nu(x)+g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1+\mu_2)}\right)^{p(x)}dx\le 1\quad\text{and}\quad \operatorname*{ess-sup}_{x\in\Omega_\infty}\frac{|f_\nu(x)+g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1+\mu_2)}\le 1. \tag{8}$$

We first prove the second part of (8). First we observe, that (6) and (7) imply, that

$$|f_\nu(x)|\le \mu_1\lambda_\nu^{1/q(x)}\quad\text{and}\quad |g_\nu(x)|\le \mu_2\Lambda_\nu^{1/q(x)}$$

holds for almost every $x\in\Omega_\infty$. Using $q(x)\ge 1$, and Hölder's inequality in the form

$$\frac{\mu_1\lambda_\nu^{1/q(x)}+\mu_2\Lambda_\nu^{1/q(x)}}{\mu_1+\mu_2}\le\left(\frac{\mu_1\lambda_\nu+\mu_2\Lambda_\nu}{\mu_1+\mu_2}\right)^{1/q(x)},$$

we get

$$\frac{|f_\nu(x)+g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1+\mu_2)}\le 1.$$

If $q(x)=\infty$, only notational changes are necessary.

Next we prove the first part of (8). Let $1\le q(x)\le p(x)<\infty$ for almost all $x\in\Omega_0$. Then we use Hölder's inequality in the form

$$\begin{aligned}&F_\nu(x)^{1/p(x)}\lambda_\nu^{1/q(x)}\mu_1+G_\nu(x)^{1/p(x)}\Lambda_\nu^{1/q(x)}\mu_2\\&\quad\le(\mu_1+\mu_2)^{1-1/q(x)}(\mu_1\lambda_\nu+\mu_2\Lambda_\nu)^{1/q(x)-1/p(x)}(F_\nu(x)\lambda_\nu\mu_1+G_\nu(x)\Lambda_\nu\mu_2)^{1/p(x)}.\end{aligned}\tag{9}$$

If $1/p(x)+1/q(x)\le 1$ for almost every $x\in\Omega_0$, then we replace (9) by

$$\begin{aligned}&F_\nu(x)^{1/p(x)}\lambda_\nu^{1/q(x)}\mu_1+G_\nu(x)^{1/p(x)}\Lambda_\nu^{1/q(x)}\mu_2\\&\quad\le(\mu_1+\mu_2)^{1-1/p(x)-1/q(x)}(\mu_1\lambda_\nu+\mu_2\Lambda_\nu)^{1/q(x)}(F_\nu(x)\mu_1+G_\nu(x)\mu_2)^{1/p(x)}.\end{aligned}\tag{10}$$

Using (9), we may further continue

$$\begin{aligned}&\int_{\Omega_0}\left(\frac{|f_\nu(x)+g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1+\mu_2)}\right)^{p(x)}dx\\&=\int_{\Omega_0}\left(\frac{F_\nu(x)^{1/p(x)}\lambda_\nu^{1/q(x)}\mu_1+G_\nu(x)^{1/p(x)}\Lambda_\nu^{1/q(x)}\mu_2}{\mu_1+\mu_2}\right)^{p(x)}\cdot\left(\frac{\mu_1\lambda_\nu+\mu_2\Lambda_\nu}{\mu_1+\mu_2}\right)^{-\frac{p(x)}{q(x)}}dx\\&\le\int_{\Omega_0}\frac{F_\nu(x)\lambda_\nu\mu_1+G_\nu(x)\Lambda_\nu\mu_2}{\mu_1\lambda_\nu+\mu_2\Lambda_\nu}dx\\&=\frac{\mu_1\lambda_\nu}{\mu_1\lambda_\nu+\mu_2\Lambda_\nu}\int_{\Omega_0}F_\nu(x)dx+\frac{\mu_2\Lambda_\nu}{\mu_1\lambda_\nu+\mu_2\Lambda_\nu}\int_{\Omega_0}G_\nu(x)dx\le 1,\end{aligned}$$

where we used also (6) and (7). If we start with (10) instead, we proceed in the following way

$$\int_{\Omega_0}\left(\frac{|f_\nu(x)+g_\nu(x)|}{A_\nu^{1/q(x)}(\mu_1+\mu_2)}\right)^{p(x)}dx$$

$$=\int_{\Omega_0}\left(\frac{F_\nu(x)^{1/p(x)}\lambda_\nu^{1/q(x)}\mu_1+G_\nu(x)^{1/p(x)}\Lambda_\nu^{1/q(x)}\mu_2}{\mu_1+\mu_2}\right)^{p(x)}\cdot\left(\frac{\mu_1\lambda_\nu+\mu_2\Lambda_\nu}{\mu_1+\mu_2}\right)^{-\frac{p(x)}{q(x)}}dx$$

$$\leq\int_{\Omega_0}\frac{F_\nu(x)\mu_1+G_\nu(x)\mu_2}{\mu_1+\mu_2}dx=\frac{\mu_1}{\mu_1+\mu_2}\int_{\Omega_0}F_\nu(x)dx+\frac{\mu_2}{\mu_1+\mu_2}\int_{\Omega_0}G_\nu(x)dx\leq 1.$$

In both cases, this finishes the proof of (8). □

2.2 Counterexample

Theorem 2. *There exist functions $p,q\in\mathcal{P}(\mathbb{R}^n)$ with $\inf_{x\in\mathbb{R}^n}p(x)\geq 1$ and $\inf_{x\in\mathbb{R}^n}q(x)\geq 1$ such that $\|\cdot|\ell_{q(\cdot)}(L_{p(\cdot)})\|$ does not satisfy the triangle inequality.*

Proof. Let $Q_0,Q_1\subset\mathbb{R}^n$ be two disjoint unit cubes, let $p(x):=1$ everywhere on $\mathbb{R}^n$ and put $q(x):=\infty$ for $x\in Q_1$ and $q(x):=1$ for $x\notin Q_1$. Let $f_1=\chi_{Q_0}$ and $f_2=\chi_{Q_1}$. Finally, we put $f=(f_1,f_2,0,\dots)$ and $g=(f_2,f_1,0,\dots)$.

We calculate for every $L>0$ fixed

$$\inf\left\{\lambda_1>0:\varrho_{p(\cdot)}\left(\frac{f_1(x)}{\lambda_1^{1/q(x)}L}\right)\leq 1\right\}=\inf\left\{\lambda_1>0:\frac{1}{\lambda_1 L}\leq 1\right\}=1/L$$

and

$$\inf\left\{\lambda_2>0:\varrho_{p(\cdot)}\left(\frac{f_2(x)}{\lambda_2^{1/q(x)}L}\right)\leq 1\right\}=\inf\left\{\lambda_2>0:\frac{1}{L}\leq 1\right\}.$$

If $L\geq 1$, then the last expression is equal to zero, otherwise it is equal to ∞.

We obtain

$$\|f|\ell_{q(\cdot)}(L_{p(\cdot)})\|=\inf\{L>0:\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f/L)\leq 1\}=\inf\{L>0:1/L+0\leq 1\}=1$$

and the same is true also for $\|g|\ell_{q(\cdot)}(L_{p(\cdot)})\|$. It is therefore enough to show, that $\|f+g|\ell_{q(\cdot)}(L_{p(\cdot)})\|>2$.

Using the calculation

$$\inf\left\{\lambda>0:\varrho_{p(\cdot)}\left(\frac{f_1(x)+f_2(x)}{L\cdot\lambda^{1/q(x)}}\right)\leq 1\right\}=\inf\left\{\lambda>0:\int_{Q_0}\frac{1}{L\cdot\lambda}+\int_{Q_1}\frac{1}{L}\leq 1\right\}$$
$$=\inf\left\{\lambda>0:\frac{1}{L\cdot\lambda}+\frac{1}{L}\leq 1\right\}=\frac{1}{L-1},$$

which holds for every $L>1$ fixed, we get

$$\|f+g|\ell_{q(\cdot)}(L_{p(\cdot)})\|=\inf\left\{L>0:\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\left(\frac{f+g}{L}\right)\leq 1\right\}$$
$$=\inf\left\{L>0:2\inf\left\{\lambda>0:\varrho_{p(\cdot)}\left(\frac{f_1(x)+f_2(x)}{L\cdot\lambda^{1/q(x)}}\right)\leq 1\right\}\leq 1\right\}$$
$$=\inf\left\{L>1:2\cdot\frac{1}{L-1}\leq 1\right\}=3.$$

□

Remark 1. Let us observe, that $1 \leq q(x) \leq p(x) \leq \infty$ holds for $x \in Q_0$ and $1/p(x) + 1/q(x) \leq 1$ is true for $x \in Q_1$. It is therefore necessary to interprete the assumptions of Theorem 1 in a correct way, namely that one of the conditions of Theorem 1 holds for (almost) all $x \in \mathbb{R}^n$. This is not to be confused with the statement, that for (almost) every $x \in \mathbb{R}^n$ at least one of the conditions is satisfied, which is not sufficient.

Remark 2. A similar calculation (which we shall not repeat in detail) shows, that one may also put $q(x) := q_0$ large enough for $x \in Q_1$ to obtain an counterexample. Hence there is nothing special about the infinite value of q and the same counterexample may be reproduced with uniformly bounded exponents $p, q \in \mathcal{P}(\mathbb{R}^n)$.

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