2-microlocal Besov spaces

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Summary. We introduce 2-microlocal Besov spaces which generalize the 2-microlocal spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ by Bony. We give a unified Fourier-analytic approach to define generalized 2-microlocal Besov spaces and we present a wavelet characterization of them. Wavelets provide a powerful tool for studying global and local regularity properties of functions.
Further, we prove a characterization with wavelets for the local version of the 2-microlocal Besov spaces and we give first connections and generalizations to local regularity theory.

Key words: Besov spaces, 2-microlocal spaces, wavelet analysis
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1 Introduction & preliminaries

In this paper we introduce 2-microlocal Besov spaces which generalize the 2-microlocal spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ introduced by Bony [4] and Jaffard [8] in two directions. For these spaces, which we call $B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w})$, we give a characterization with wavelets and use this result to describe the local 2-microlocal Besov spaces.
2-microlocal spaces initially appeared in the book of Peetre [21] and have been studied by Bony [4] in the context of non-linear hyperbolic equations and were widely elaborated by Jaffard & Meyer [9]. In [16] Lévy Véhel & Seuret developed the 2-microlocal formalism, which is similar to the multifractal formalism. It turned out, that the 2-microlocal spaces are an useful tool to measure local regularity of functions. The approach is Fourier analytic and the spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ are defined by size estimates of the Littlewood-Paley decomposition.
More precisely, let φ_0 be a positive function from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of infinitely differentiable and rapidly decreasing functions with

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$$\varphi_0(x) = \begin{cases} 1 \ , & \text{if } |x| \le 1 \ , \\ 0 \ , & \text{if } |x| \ge 2 \ . \end{cases} \tag{1}$$

We set $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ and define $\varphi_j(x) = \varphi(2^{-j}x)$ for $j = 1, 2, \ldots$. Then we have $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ and $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is called a smooth dyadic resolution of unity.
The dual space of $\mathcal{S}(\mathbb{R}^n)$ is the space of tempered distributions which we denote by $\mathcal{S}'(\mathbb{R}^n)$. By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the Fourier transform and its inverse on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, respectively. We will use also the symbols $\hat{f}$ and $f^\vee$ for $\mathcal{F}f$ and $\mathcal{F}^{-1}f$.
For $f \in \mathcal{S}'(\mathbb{R}^n)$ and a smooth resolution of unity $\{\varphi_j\}_{j\in\mathbb{N}_0}$ we have the fundamental decomposition

$$f = \sum_{j=0}^{\infty} (\varphi_j \hat{f})^\vee \ , \text{ convergence in } \mathcal{S}'(\mathbb{R}^n).$$

A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ does belong to the space $C^{s,s'}_{x_0}(\mathbb{R}^n)$, if the estimates

$$|(\varphi_j \hat{f})^\vee(x)| \le c2^{-js}(1 + 2^j|x - x_0|)^{-s'} \tag{2}$$

hold for all $x \in \mathbb{R}^n$ and all $j \in \mathbb{N}_0$. We can reformulate (2) as

$$\sup_{x\in\mathbb{R}^n} w_j(x)|(\varphi_j \hat{f})^\vee(x)| < c2^{-js} \ , \tag{3}$$

with the weight sequence

$$w_j(x) = (1 + 2^j|x - x_0|)^{s'} \ . \tag{4}$$

With the same weight functions w_j from (4) the spaces $H^{s,s'}_{x_0}(\mathbb{R}^n)$ are defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$c_j^2 = \int_{\mathbb{R}^n} w_j^2(x)|(\varphi_j \hat{f})^\vee(x)|^2 dx \qquad \text{and} \ \sum_{j\in\mathbb{N}_0} 2^{2js} c_j^2 < \infty \ . \tag{5}$$

Spaces of this type have been introduced by Bony [4]. A characterization of $C^{s,s'}_{x_0}(\mathbb{R}^n)$ by wavelets has been given by Jaffard in [8]. Wavelets provide a powerful tool to study regularity properties of functions, as can be seen in Lévy Véhel & Seuret [16].
They used the wavelet characterization of $C^{s,s'}_{x_0}(\mathbb{R}^n)$ and developed the 2-microlocal formalism. It turned out, that 2-microlocal spaces provide a fine way of measuring local smoothness of distributions. Many regularity exponents, as the local and pointwise Hölder exponent, the chirp exponent, the oscillating exponent and the weak scaling exponent, can be derived just by calculating the 2-microlocal domain (see [17] and [16] for details). This 2-microlocal domain is the set

$$E(f,x_0)=\left\{(s,s')\in\mathbb{R}^2 : f \text{ belongs to } C^{s,s'}_{x_0}(\mathbb{R}^n) \text{ locally around } x_0\right\} .$$

We will introduce a more general 2-microlocal domain in Section 3 based on the 2-microlocal Besov spaces $B^{s,mloc}_{p,q}(\mathbb{R}^n,\boldsymbol{w})$ which are defined below.

Conditions (3) and (5) suggest to consider $C^{s,s'}_{x_0}(\mathbb{R}^n)$ and $H^{s,s'}_{x_0}(\mathbb{R}^n)$ as a kind of weighted Besov spaces. In general a distribution $f\in\mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_{p,q}(\mathbb{R}^n,w)$ for $s\in\mathbb{R}$ and $0<p,q\le\infty$, if the (quasi-)norm of f satisfies

$$\left\| f \,|\, B^s_{p,q}(\mathbb{R}^n,w)\right\| = \left(\sum_{j=0}^{\infty} 2^{jsq}\left\|(\varphi_j\hat{f})^\vee \,\Big|\, L_p(\mathbb{R}^n,w)\right\|^q\right)^{1/q} < \infty \ , \qquad (6)$$

where w is an admissible weight function (see [6]). Here, $L_p(\mathbb{R}^n)$ denotes the usual Lebesgue space and its weighted version $L_p(\mathbb{R}^n,w)$ is normed by

$$\| f \,|\, L_p(\mathbb{R}^n,w)\| = \| wf \,|\, L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |w(x)f(x)|^p dx\right)^{1/p} . \qquad (7)$$

Now, it becomes obvious how to modify the definition of the Besov space norm (6) to obtain generalized 2-microlocal Besov spaces. We replace w in (6) by the special weights w_j from (4), depending also on $j\in\mathbb{N}_0$.

We will deal with a further generalization with respect to the weight sequence. Instead of the weights from (4) we introduce the notion of admissible weight sequences.

Definition 1 (Admissible weight sequence). *Let $\alpha\ge 0$ and let $\alpha_1,\alpha_2\in\mathbb{R}$, $\alpha_1\le\alpha_2$. A sequence of non-negative measurable functions $\boldsymbol{w}=\{w_j\}_{j=0}^\infty$ belongs to the class $\mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ if, and only if,*

(i) There exists a constant $\mathrm{C}>0$ such that

$$0<w_j(x)\le \mathrm{C}w_j(y)\left(1+2^j|x-y|\right)^\alpha \quad \text{for all } j\in\mathbb{N}_0 \text{ and all } x,y\in\mathbb{R}^n.$$

(ii) For all $j\in\mathbb{N}_0$ and all $x\in\mathbb{R}^n$ we have

$$2^{\alpha_1}w_j(x)\le w_{j+1}(x)\le 2^{\alpha_2}w_j(x) \ .$$

Such a system $\{w_j\}_{j=0}^\infty\in\mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ is called admissible weight sequence.

For $U\subset\mathbb{R}^n$ we denote $\mathrm{dist}(x,U)=\inf_{y\in U}|x-y|$ and we define for $s'\in\mathbb{R}$ the 2-microlocal weights by

$$w_j(x)=(1+2^j\,\mathrm{dist}(x,U))^{s'} \ . \qquad (8)$$

These weights are an admissible weight sequence with $\alpha_1=\min(0,s')$, $\alpha_2=\max(0,s')$ and $\alpha=|s'|$. Note that for $U=\{x_0\}$ we get the 2-microlocal weights (4) from the beginning. Further examples of admissible weight sequences can be found in [11].

Now, we are able to give the definition of generalized 2-microlocal Besov spaces.

Definition 2. *Let* $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *and let* $\{\varphi_j\}_{j\in\mathbb{N}_0}$ *be a smooth resolution of unity. Further, let* $0 < p, q \le \infty$ *and* $s \in \mathbb{R}$*, then*

$$B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \left\| f \right| B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w}) \right\| < \infty \right\} \quad , \textit{ where}$$

$$\left\| f \right| B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w}) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| (\varphi_j \hat{f})^{\vee} \right| L_p(\mathbb{R}^n, w_j) \right\|^q \right)^{1/q} .$$

These spaces have been introduced in [11]. Using a Fourier multiplier theorem for weighted Lebesgue spaces of entire analytic functions ([22, Theorem 1.7.5]) it is easy to show that the definition is independent of the chosen resolution of unity (see Theorem 2.13 in [11]).
If $w_j(x) = 1$ for $j \in \mathbb{N}_0$, then we obtain the usual Besov spaces from (6), studied in detail by Triebel in [24] and [25]. If we set

$$w_j(x) = w_0(x)$$

for all $j \in \mathbb{N}_0$, then we derive weighted Besov spaces $B^s_{p,q}(\mathbb{R}^n, w_0)$ which were studied in [6, Chapter 4].
Regarding the 2-microlocal weight sequence

$$w_j(x) = (1 + 2^j|x - x_0|)^{s'} ,$$

we get for $p = q = \infty$ the spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ introduced by Jaffard [8] and for $p = q = 2$ we obtain the spaces $H^{s,s'}_{x_0}(\mathbb{R}^n)$ introduced by Bony [4]. With these weight functions, Xu studied in [28] 2-microlocal Besov spaces with $1 \le p, q \le \infty$ and in [18] Meyer & Xu used these spaces to characterize chirps by means of their wavelet transforms.
Using as admissible weight sequence the weights from (8) with open $U \subset \mathbb{R}^n$, Moritoh & Yamada introduced in [19] 2-microlocal Besov spaces of homogeneous type and studied local properties of functions.
Taking $w_j(x) = \sigma_j$ for $j \in \mathbb{N}_0$ where $\sigma_j \in \mathbb{R}$ satisfies $c_1\sigma_j \le \sigma_{j+1} \le c_2\sigma_j$ for some $c_1, c_2 > 0$, we derive so called Besov spaces of generalized smoothness introduced by Kalyabin [10] and studied in [7] and [20]. More generally, we can set

$$w_j(x) = 2^{js(x)}$$

with suitable conditions on $s(x) : \mathbb{R}^n \to \mathbb{R}$ ([14]) and we obtain spaces of variable smoothness introduced by Underberger & Bokobza [26] and Beauzamy [2] and more recent results are due to Leopold [15] and Besov [3].
The above definition with weights satisfying Definition 1 was given in [11] by Kempka and characterizations by local means, atoms and wavelets have been established ([11], [13]). Moreover, there exists also a characterization by differences of $B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w})$ proved by Besov in [3].
To this end, we define by $\Delta_h f(x) = f(x+h) - f(x)$ and $\Delta^M_h = \Delta^{M-1}_h \Delta_h$ the

iterated differences for $x, h \in \mathbb{R}^n$ and $M \in \mathbb{N}$. Two norms $(\|\cdot\|_1, \|\cdot\|_2)$ are called equivalent on a space X, if there exists a constant $c > 0$ such that

$$\frac{1}{c}\|x\|_1 \le \|x\|_2 \le c\|x\|_1 \quad \text{for all } x \in X.$$

Proposition 1 (Besov 2003). *Let* $1 < p, q \le \infty$, $s > 0$ *and* $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$. *If* $M > s + \alpha_2$, *then*

$$\left(\sum_{k=1}^{\infty} 2^{ks} \sup_{|h|\le 1} \left\| w_k \Delta^M_{2^{-k}h} f \right| L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} + \| w_0 f | L_p(\mathbb{R}^n)\|$$

is an equivalent norm on $B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w})$.

This corresponds to the time domain characterization of the local version of $C^{s,s'}_{x_0}(\mathbb{R}^n)$ presented in [23] by Seuret & Lévy Véhel.
Another approach, which is not covered by Definition 2, is to generalize $H^{s,s'}_{x_0}(\mathbb{R}^n)$ as weighted Triebel-Lizorkin spaces. This has been done by Andersson in [1] for the 2-microlocal weights from (4). In a more general context these spaces have been studied with admissible weight sequences from Definition 1 in [14] and local means characterizations have been established.

In the next section we present an adapted wavelet characterization based on Daubechies wavelets for $B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w})$ with weights from (8). Section 3 deals with the local version of these spaces and we use the results from the previous section to describe them with wavelet decompositions as in [9, Proposition 1.4] and [16, Theorem 1] for $U = \{x_0\}$ and $p = q = 2$ or $p = q = \infty$.
Although we do not develop a full regularity theory of functions as in [16] our results seem to be promising for further research.

2 Characterization with wavelets

In this section we will present a wavelet characterization for $B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w})$ with the weight sequence from (8). In comparison to $C^{s,s'}_{x_0}(\mathbb{R}^n)$ we will denote them by $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$.
The most important characterization of the local spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ is due to the wavelet characterization. To this end, we have to give a modified version of the wavelet characterization in Theorem 4 in [13]. We adopt the notation from [25, 4.2.1]. For sufficiently large $k \in \mathbb{N}_0$, let us assume that

$$\psi_M, \psi_F \in C^k(\mathbb{R}) \tag{9}$$

are real, compactly supported Daubechies wavelets (see [5],[27]) with

$$\int_{\mathbb{R}} x^{\beta} \psi_M(x) dx = 0 \qquad \text{for } |\beta| < k \tag{10}$$

and $\operatorname{supp}\psi_M, \operatorname{supp}\psi_F \subset B_{2^J}(0)$, with $J \in \mathbb{N}$. Here, $B_r(x)$ denotes the open ball around $x \in \mathbb{R}^n$ with radius $r > 0$. Let $l \in \mathbb{N}_0$ then

$$G = G^{l,l} = \{F, M\}^n \qquad \text{and} \qquad G^{\nu,l} = \{F, M\}^{n*} \quad \text{for } \nu > l,$$

where the * indicates that at least one G_i of $G = (G_1, \ldots, G_n) \in \{F, M\}^{n*}$ must be an M. It is well known that $\{\Psi^{\nu,l}_{Gm} : \nu \geq l,\, G \in G^{\nu,l} \text{ and } m \in \mathbb{Z}^n\}$ is an orthonormal basis of $L_2(\mathbb{R}^n)$ for fixed $l \in \mathbb{N}_0$ with

$$\Psi^{\nu,l}_{Gm}(x) = 2^{\nu\frac{n}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^\nu x_r - m_r) \quad \text{where } G = (G_1, \ldots, G_n) \in G^{\nu,l}.$$

We have to adapt our sequence spaces to the new situation. A sequence of complex-valued numbers $\{\lambda^{\nu,l}_{Gm}\}$ belongs to $b^{s,s'}_{p,q;l}(U)$ if, and only if,

$$\left\| \lambda |\, b^{s,s'}_{p,q;l}(U) \right\| = \left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G \in G^{\nu,l}} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^{\nu,l}_{Gm}|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{s'p} \right)^{q/p} \right)^{1/q} < \infty \,.$$

We introduce the number $\sigma_p = \max(0, n(1/p - 1))$ which is zero if $p \geq 1$. By unconditional convergence of a sum we mean that each rearrangement of the sum converges to the same limit. The next corollary follows from Theorem 4 in [13].

Corollary 1. *Let $U \subset \mathbb{R}^n$ bounded, $s, s' \in \mathbb{R}$ and $l \in \mathbb{N}_0$. Further, let $0 < p, q \leq \infty$ and*

$$k > \max(\sigma_p - s - \min(0, s'), s + \max(0, s')) \tag{11}$$

in (9) and (10). Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$ if, and only if, it can be represented as

$$f = \sum_{\nu=l}^{\infty} \sum_{G \in G^{\nu,l}} \sum_{m \in \mathbb{Z}^n} \lambda^{\nu,l}_{Gm} 2^{-\nu\frac{n}{2}} \Psi^{\nu,l}_{Gm} \qquad \text{with } \lambda \in b^{s,s'}_{p,q;l}(U)\ , \tag{12}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any $B^{t,t'}_{p,q}(\mathbb{R}^n, U)$ with $t < s$ and $t' < s'$. The representation (12) is unique,

$$\lambda^{\nu,l}_{Gm} = \lambda^{\nu,l}_{Gm}(f) = 2^{\nu\frac{n}{2}} \left\langle f, \Psi^{\nu,l}_{Gm} \right\rangle \tag{13}$$

and

$$I : f \mapsto \{2^{\nu\frac{n}{2}} \left\langle f, \Psi^{\nu,l}_{Gm} \right\rangle\} \tag{14}$$

is an isomorphic map from $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$ onto $b^{s,s'}_{p,q;l}(U)$. Moreover, if in addition $\max(p, q) < \infty$ then $\{\Psi^{\nu,l}_{Gm}\}$ is in unconditional basis in $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$.

The advantage of this representation with additional index $l \in \mathbb{N}_0$ is that the size of the support of the wavelets on the zero level $\nu = l$ is $\operatorname{supp} \Psi^{l,l}_{Gm} \subset B_{2^{J-l}}(2^{-l}m)$ and can be minimized by taking large $l \in \mathbb{N}_0$.

Remark 1. We assume in the following that the Daubechies wavelets have enough regularity, which means $k > \max(\sigma_p - s - \min(0, s'), s + \max(0, s'))$. Note that in the case $p \geq 1$ this means $k > \max(|s|, |s + s'|)$.

3 The local spaces $B^{s,s'}_{p,q}(U)^{loc}$

This section is devoted to the study of the local spaces $B^{s,s'}_{p,q}(U)^{loc}$. They are an appropriate instrument for measuring local regularity of functions as has been done intensively by Jaffard & Meyer, Seuret & Lévy Véhel and many others ([9], [16], [18]). We would like to point out some connections to the known case, $p = q = \infty$ and $U = \{x_0\}$, and give first results.
For the rest of the paper we fix $U \subset \mathbb{R}^n$ as a compact subset and $s, s' \in \mathbb{R}$ and $0 < p, q \leq \infty$ are arbitrary but fixed numbers.

3.1 Definition and wavelet characterization

In this subsection we define the local version of $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$ for compact $U \subset \mathbb{R}^n$ and give a characterization by wavelets for them.

Definition 3. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then f belongs to the local space $B^{s,s'}_{p,q}(U)^{loc}$ if there exists an open neighborhood $V_0 \supset U$ and $g \in B^{s,s'}_{p,q}(\mathbb{R}^n, U)$ globally such that $f = g$ on V_0.*

From a pointwise multiplier statement for the global spaces $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$ (Theorem 4.10 in [11]) we obtain the following.

Lemma 1. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B^{s,s'}_{p,q}(U)^{loc}$ if, and only if, there exists an open neighborhood $V_0 \supset U$ and $\varphi \in C^\infty_0(\mathbb{R}^n)$ with $\varphi(x) = 1$ on V_0 and $\varphi f \in B^{s,s'}_{p,q}(\mathbb{R}^n, U)$.*

Now, we are able to characterize the local spaces $B^{s,s'}_{p,q}(U)^{loc}$ in terms of wavelets.

Theorem 1. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then f belongs to $B^{s,s'}_{p,q}(U)^{loc}$ if, and only if, there exists an $l \in \mathbb{N}_0$ and an $A > 0$ with*

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G \in G^{\nu,l}} \left(\sum_{m \in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{s'p} \right)^{q/p} \right)^{1/q} < \infty \,, \tag{15}$$

where

$$U_\nu = \{m \in \mathbb{Z}^n : \operatorname{dist}(2^{-\nu}m, U) \le A\} \quad \textit{and} \quad \lambda^{\nu,l}_{Gm}(f) = 2^{\nu n/2}\left\langle f, \Psi^{\nu,l}_{Gm}\right\rangle .$$

Proof. First Step: We have $f \in B^{s,s'}_{p,q}(U)^{loc}$, which means that we can find open sets V_0, V such that $U \subset V_0 \subset V$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi(x) = 1$ on V_0, $\operatorname{supp}\varphi \subset V$ and $\varphi f \in B^{s,s'}_{p,q}(\mathbb{R}^n, U)$. We choose a number, $-h \in \mathbb{N}_0$ such that $U_{2^h} \subset V_0$, where $U_{2^h} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, U) \le 2^h\}$. We would like to take these $\Psi^{\nu,l}_{Gm}$ which fulfill

$$\left\langle \varphi f, \Psi^{\nu,l}_{Gm}\right\rangle = \left\langle f, \Psi^{\nu,l}_{Gm}\right\rangle , \tag{16}$$

which means that $\operatorname{supp}\Psi^{\nu,l}_{Gm} \subset U_{2^h} \subset V_0$. This is fulfilled if $\operatorname{dist}(2^{-j}m, U) \le 2^h - 2^{J-\nu}$. To have a positive number on the right hand side we have to demand $\nu > J - h$. Now, we fix $l = J - h + 1$ and $A > 0$ by $A = 2^h - 2^{J-l}$. From Corollary 1 we derive that

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G\in G^{\nu,l}} \left(\sum_{m\in\mathbb{Z}^n} |\lambda^{\nu,l}_{Gm}(\varphi f)|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{s'p}\right)^{q/p}\right)^{1/q} < \infty$$

and that finally gives us with (16) and $U_\nu = \{m \in \mathbb{Z}^n : \operatorname{dist}(2^{-\nu}m, U) \le A\}$ with $A > 0$ as above

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G\in G^{\nu,l}} \left(\sum_{m\in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1 + 2^\nu \operatorname{dist}(2^{-\nu}m, U))^{s'p}\right)^{q/p}\right)^{1/q} < \infty .$$

Second step: If we have (15) for some $l \in \mathbb{N}_0$ and $A > 0$, then we can define

$$\tilde\lambda^{\nu,l}_{Gm} = \begin{cases} \lambda^{\nu,l}_{Gm} & \text{, for } m \in U_\nu \\ 0 & \text{, otherwise.} \end{cases}$$

Then $f = \sum_{\nu,G,m} \tilde\lambda^{\nu,l}_{Gm} 2^{-\nu\frac{n}{2}} \Psi^{\nu,l}_{Gm}$ belongs to $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$ by Corollary 1 and this implies $f \in B^{s,s'}_{p,q}(U)^{loc}$. □

Remark 2. Let us emphasize that this theorem is similar to [9, Proposition 1.4] and [16, Theorem 1] in the cases $p = q = \infty$, $p = q = 2$ and $U = \{x_0\}$.

3.2 Embeddings

The aim of this subsection is to present some embedding theorems for the local spaces. These embeddings are well known in the case $p = q = \infty$ and $U = \{x_0\}$.

Lemma 2. *Let* $f \in B^{s,s'}_{p,q}(U)^{loc}$*, then* f *belongs to* $B^{s-\varepsilon,s'+\varepsilon}_{pq}(U)^{loc}$ *for all* $\varepsilon > 0$.

The proof is a simple application of the theorem above. More generally we can prove the following embedding.

Theorem 2.

$$B^{s,s'}_{p,q}(U)^{loc} \hookrightarrow B^{t,t'}_{p,q}(U)^{loc} \quad \textit{if, and only if,} \quad t \le s \textit{ and } t + t' \le s + s' \ .$$

Proof. The sufficiency of the conditions with respect to the parameters $s, t, s', t' \in \mathbb{R}$ is proved again using Theorem 1. To get the necessity we have to be more careful. The embedding is equivalent to the fact that we can find $l \in \mathbb{N}_0$, $A > 0$ and $c > 0$ such that

$$2^{(t-s)\nu} \le c(1 + 2^{\nu}\,\mathrm{dist}(2^{-\nu}m, U))^{s'-t'} \quad \text{holds for all } \nu \ge l \text{ and } m \in U_{\nu}. \quad (17)$$

We have to distinguish two cases. First, we assume that $s < t$, then for $\nu \ge l$ large enough, we can find $m_{\nu} \in U_{\nu}$ with $\mathrm{dist}(2^{-\nu}m_{\nu}, U) \sim 2^{-\nu}$. This implies that the left hand side of (17) is increasing in ν. But, the right hand side is independent of ν which is a contradiction to (17).
In the second case we assume that $t + t' > s + s'$. Then we take for all $\nu \ge l$ an $m_{\nu} \in U_{\nu}$ with $\mathrm{dist}(2^{-\nu}m_{\nu}, U) \sim A$. We can estimate the right hand side of (17) by

$$(1 + 2^{\nu}\,\mathrm{dist}(2^{-\nu}m_{\nu}, U))^{s'-t'} \le c2^{\nu(s'-t')} \quad \text{where } c > 0 \text{ is independent of } \nu.$$

This gives us a contradiction to (17), because there does not exist $c > 0$ with $2^{\nu(t-s)} \le c2^{\nu(s'-t')}$ for all $\nu \ge l$. □

Remark 3. This embedding theorem is in contrast to the global spaces, where we have (Remark 2.3.4 in [12])

$$B^{s,s'}_{p,q}(\mathbb{R}^n, U) \hookrightarrow B^{t,t'}_{p,q}(\mathbb{R}^n, U) \quad \text{if, and only if,} \quad t \le s \text{ and } t' \le s' \ .$$

These results are well known in the case of the local spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ ([17, Corollary III/3.4]). Moreover, this theorem is the starting point for the definition of the so-called 2-microlocal frontier, see [17, III.5] and [16, Chapter 2].

3.3 The 2-microlocal domain

Similary as in [17] we give in this subsection a generalized approach to define a 2-microlocal domain for a given function $f \in \mathcal{S}'(\mathbb{R}^n)$.

Definition 4. *Let* $f \in \mathcal{S}'(\mathbb{R}^n)$*, then for fixed* $0 < p, q \le \infty$

$$E_{p,q}(f, U) = \{(s, s') \in \mathbb{R}^2 : f \in B^{s,s'}_{p,q}(U)^{loc}\}$$

defines the 2-microlocal domain.

We have generalized the 2-microlocal domain from [17] and [16] where case $p = q = \infty$ has been considered. We get from the embedding Theorem 2 the following.

Lemma 3. *Let* $(s, s') \in E_{p,q}(f, U)$ *and let*

$$t \leq s \quad \text{and} \quad t + t' \leq s + s' \ ,$$

then $(t, t') \in E_{p,q}(f, U)$.

Moreover, an easy application of Theorem 1 shows that this domain is convex.

Lemma 4. *The 2-microlocal domain is convex. This means if* $(s, s') \in E_{p,q}(f, U)$ *and* $(t, t') \in E_{p,q}(f, U)$ *then* $(\lambda s + (1-\lambda)t, \lambda s' + (1-\lambda)t') \in E_{p,q}(f, U)$ *for all* $\lambda \in [0, 1]$.

Remark 4. This 2-microlocal domain clearly gives us new information about the local regularity of functions (distributions). As a first example we take the delta distribution and $U \subset \mathbb{R}^n$ compact with $0 \in U$. Then we have for $0 < q < \infty$

$$\delta \in B^{s,s'}_{p,q}(U)^{loc} \Leftrightarrow s < \frac{n}{p} - n$$

and for $q = \infty$

$$\delta \in B^{s,s'}_{p,\infty}(U)^{loc} \Leftrightarrow s \leq \frac{n}{p} - n \ .$$

Hence, one easily recognizes the role played by the parameter p and, less important, q.

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